Coupling in Design and Robust Control Optimization

Sulaiman F. Alyaqout, Panos Y. Papalambros, and A. Galip Ulsoy

***Abstract*— System performance can significantly benefit from optimally integrating the design and control of engineering systems. As part of an effort to improve the robustness properties of systems, the present article introduces a general approach that combines design with robust control and investigates the properties of coupling between them. This integrated design and robust control approach leads to a coupled optimization problem that is often difficult to solve. This article introduces an explicit measure of coupling between the design and robust control optimization problems. The effect of parameters (i.e., level of uncertainty, objective weights, design and control model parameters) on this coupling is then studied. Varying such parameters leads to establishing a relationship between coupling and robustness of the control system. This link between coupling and robustness can be utilized to aid the designer in assessing the parameters' influence on coupling and robustness. Results show that the coupling strength between design and robust control tends to increase as the applied level of uncertainty increases.**

***Index Terms*— coupling, design, optimization, optimal control, robust control, robustness**

I. INTRODUCTION

Design and control integration has been considered in recent research on vehicle suspensions [1], [2], flexible structures [3], [4], [5], and electric motors [6], [7]. These studies have shown that traditional sequential strategies, that involve optimizing the design first and then optimizing the control, can lead to inferior designs compared to system-level strategies [8], [9], [10]. Such system-level strategies improve performance by guaranteeing optimality of the combined design and control optimization problem.

Solving combined design and control optimization problems has received substantial attention from researchers. Fathy [10] introduced a system-level approach to solving the combined design and control optimization problem and compared his system-level approach to several other optimization strategies. Tanaka [11] proposed a general formulation to the combined structure and control optimization problem and reduced the formulation to a Bilinear matrix inequality (BMI) using a descriptor form; he also proposed an algorithm to solve the optimization problem. In contrast, iterative redesign strategies proposed by Skelton [12], [13] have the advantage of guaranteeing a locally convergent solution to the optimization problem.

In contrast, few researchers considered integrating design and control with the aim of enhancing the robustness properties of the control system. Initial contribution to this area include utilizing the numerical tool (ASTROS) in the structure and robust control design of a fighter aircraft with tip-missile inertial uncertainty [3]. While, Asada [5] proposed an optimization method with structural design variables that minimizes the robustness margin (structured singular value) of a closed-loop system and still maintains the validity of a reduced order-model.

Several authors have examined the coupling between design and control optimization problems [4], [14], [15]. Reyer [8] and Fathy [10] proposed the use of optimality conditions to characterize coupling. They defined coupling relative to the solution method, for example, by comparing the optimality conditions of a sequential solution strategy with the optimality conditions of the undecomposed system optimization problem. Hence, this coupling has been shown to have a direct effect on determining the system optimum. For a different interpretation of coupling, Brusher [16] presented a theoretical study of the coupling in terms of the size of set of models from which satisfactory controllers may be derived.

Robust control methods assume that the design variables are considered as parameters and should not be modified during the controller design process. Therefore, the effect of the design variables in improving the robustness of the control system has not been studied. In addition, Fathy's [10] formulation used for coupling characterization does not incorporate uncertainty. As a result, previous contributions have not studied explicitly the relationship between coupling and robustness. Moreover, Fathy's [10] coupling was derived based on the assumption of differentiability. Here, we extend this coupling to deal with discontinuous functions by utilizing tools from nonsmooth optimization theory.

This article examines how the coupling between design and robust control can be linked to the robustness of the control system. This link between coupling and robustness can be utilized for aiding the designer in assessing the parameters' influence on coupling and robustness. Section II poses the system problem under consideration, while Section III shows how the sequential strategy used to solve the system problem is defined. Section IV characterizes the coupling between design and robust control and Section V illustrates the relationship between coupling and robustness. Section VI presents an elevator example implementation for the coupling and robustness relationship. The article concludes, in Section VII, with a discussion of results and future work.

II. THE DESIGN AND ROBUST CONTROL OPTIMIZATION PROBLEM

Consider a general formulation that combines the design and robust control optimization problems. The formulation optimizes the combined design and control objective function in the presence of uncertainty (performance robustness) while maintaining constraint feasibility also in the presence of uncertainty (feasibility robustness). The design optimization problem is assumed nominal with no uncertainty associated with it; however, the control optimization problem is affected by uncertainty. This assumption represents a common industry practice where the robust design is not sought due to complexity, expense and small expected gains. But during the controller integration stage uncertainty becomes a critical issue and must be considered to satisfy the control requirements robustly.

The design and robust control (DRC) optimization problem is

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stated as follows.

$$\min_{\bar{\mathbf{x}}} \max_{\Delta\bar{\mathbf{x}}} \quad w_d F_d(\mathbf{x}_d, \mathbf{p}_d) + w_c \int_{t_i}^{t_f} f_c(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

subject to

$$\dot{\mathbf{x}}(t) = \boldsymbol{r}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c)$$
$$\mathbf{h}_d(\mathbf{x}_d, \mathbf{p}_d) = \mathbf{0} \quad , \quad \mathbf{h_c}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) = \mathbf{0}$$
$$(\bar{\mathbf{x}}, \Delta\bar{\mathbf{x}}) \in \{(\bar{\mathbf{x}}, \Delta\bar{\mathbf{x}}) \in \mathbb{R}^{n_{\bar{\mathbf{x}}}} \times \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} :$$
$$\mathbf{g}_c(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) \leq \mathbf{0},$$
$$\text{for every } \Delta\bar{\mathbf{x}} \in \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} : \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0}\}\}$$
$$\mathbf{g}_d(\mathbf{x}_d, \mathbf{p}_d) \leq \mathbf{0} \quad , \quad \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0} \tag{1a}$$

where

$$\bar{\mathbf{x}} \triangleq \left(\mathbf{x}_d^T, \mathbf{u}^T(t), \mathbf{x}^T(t)\right)^T, \Delta\bar{\mathbf{x}} \triangleq \left(\Delta\mathbf{x}_d^T, \Delta\mathbf{p}_d^T, \Delta\mathbf{p}_c^T\right)^T$$
$$\hat{\mathbf{x}}_d = f_{\mathbf{x}_d}(\mathbf{x}_d, \Delta\mathbf{x}_d) \quad , \quad \hat{\mathbf{p}}_d = f_{\mathbf{p}_d}(\mathbf{p}_d, \Delta\mathbf{p}_d) \tag{1b}$$
$$\hat{\mathbf{p}}_c = f_{\mathbf{p}_c}(\mathbf{p}_c, \Delta\mathbf{p}_c) \tag{1c}$$

and $\mathbf{x}(t) \in \mathbb{R}^n$ is the column vector of system state variables, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\mathbf{x}_d \in \mathbb{R}^{n_d}$ is the column vector of design variables, $\mathbf{p_d} \in \mathbb{R}^{n_{p_d}}$ is the vector of design parameters, $\mathbf{p}_c \in \mathbb{R}^{n_{p_c}}$ is the vector of control parameters, $\bar{\mathbf{x}} \in \mathbb{R}^{n_{\bar{\mathbf{x}}}}$ is the vector of combined optimization variables where $n_{\bar{\mathbf{x}}}{=}n_{n_d}{+}n{+}m$. In addition, $F_d \in \mathbb{R}$, $f_c(t) \in \mathbb{R}$ is the objective function representing design, control performance, respectively; $\mathbf{g}_d \in \mathbb{R}^{n_{g_d}}, \mathbf{h}_d \in \mathbb{R}^{n_{h_d}}$ represent the design constraints, while $\mathbf{g}_c \in \mathbb{R}^{n_{g_c}}, \mathbf{h}_c \in \mathbb{R}^{n_{h_c}}$ represent the control constraints. Furthermore, $(.)^T$ is vector transpose and $\triangleq$ represents a definition.

Several assumptions are associated with the design and robust control (DRC) formulation. The DRC formulation is a worst-case approach as represented by the minimax formulation in Eq. (1a). We assume real set-based uncertainty $\Delta\mathbf{x}_d \in \mathbb{R}^{n_d}$, $\Delta\mathbf{p}_d \in \mathbb{R}^{n_{p_d}}$, $\Delta\mathbf{p}_c \in \mathbb{R}^{n_{p_c}}$ resulting from variation in the design variables $\mathbf{x}_d$, the design parameters $\mathbf{p}_d$, and the control parameters $\mathbf{p}_c$, respectively. Note that in the problem statement above all the uncertainty is parametric; no model uncertainty, which often results from model reduction, is included in the DRC formulation.

Furthermore, the functions $f_{\mathbf{x}_d}$, $f_{\mathbf{p}_d}$, $f_{\mathbf{p}_c}$ with resulting variables $\hat{\mathbf{x}}_d \in \mathbb{R}^{n_d}$, $\hat{\mathbf{p}}_d \in \mathbb{R}^{n_{p_d}}$, $\hat{\mathbf{p}}_c \in \mathbb{R}^{n_{p_c}}$ defined in Eq. (1b) and Eq. (1c) represent the assumed type of uncertainty (additive, percentage) associated with design variables $\mathbf{x}_d$, design parameters $\mathbf{p}_d$, control parameters $\mathbf{p}_c$, respectively. For example, the uncertainty $\Delta\mathbf{x}_d$ can represent a 50% increase (i.e., $\Delta\mathbf{x}_d = 1.5$), while the variable $\hat{\mathbf{x}}_d$ represents the actual variable change $\hat{\mathbf{x}}_d = 1.5\mathbf{x}_d$. In addition, $\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}}$ is the vector of combined uncertainty variables where $n_{\Delta\bar{\mathbf{x}}}{=}n_{n_d}{+}n_{p_d}{+}n_{p_c}$. The $\boldsymbol{\alpha}_{\bar{\mathbf{x}}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}}$ is the level of uncertainty in design variables, design parameters, and control parameters. The $\boldsymbol{\alpha}_{\bar{\mathbf{x}}}$ and the inequality constraint $\mathbf{g}_{\bar{\mathbf{x}}}{\in}\mathbb{R}^{n_{g_{\bar{\mathbf{x}}}}}$ are used to define the uncertainty set. The final assumption is the transformation of the multi-objective optimization problem into a linear scalar substitute objective function using the design, control weights $w_d, w_c \in \mathbb{R}$, respectively. This choice of scalarization achieves the true Pareto curve only for a convex Pareto curve, but it is used here for simplicity.

There are several challenges associated with solving the DRC optimization problem. The solution requires heavy computational effort even for modestly-sized problems, since the number of design variables under uncertainty increases the complexity of the optimization problem. In addition, the DRC can be a non-convex optimization problem even if the individual design and robust control optimization problems are convex.

III. The Sequential Design and Robust Control Optimization Strategy

Some of the problems associated with solving the DRC optimization problem can be avoided by utilizing a sequential strategy under uncertainty that often yields a suboptimal solution (i.e., improved solution but not optimal). This sequential strategy under uncertainty optimizes the design problem first. Then the design solution is used in the optimization of the robust control problem, thus saving time and cost. The sequential strategy under uncertainty defined in this section provides a foundation for the work presented later in the article.

The sequential design and robust control (SDRC) strategy starts by solving the following design problem

$$\min_{\mathbf{x}_d} \quad F_d(\mathbf{x}_d, \mathbf{p}_d)$$

subject to

$$\mathbf{h}_d(\mathbf{x}_d, \mathbf{p}_d) = \mathbf{0} \quad , \quad \mathbf{g}_d(\mathbf{x}_d, \mathbf{p}_d) \leq \mathbf{0} \tag{2}$$

The design problem in Eq. (2) has no dependence on the control variables and parameters. This makes the problem easier to solve and removes the need for iteration between the design and robust control optimization problems. After solving the design problem in Eq. (2), the resulting optimal solution is used to solve the following robust control problem

$$\min_{\mathbf{x}(t),\mathbf{u}(t)} \max_{\Delta\mathbf{x}_d, \Delta\mathbf{p}_d, \Delta\mathbf{p}_c} \int_{t_i}^{tf} f_c(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

subject to

$$\dot{\mathbf{x}}(t) = \boldsymbol{r}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c)$$
$$\mathbf{h_c}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) = \mathbf{0} \ , \ \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0},$$
$$(\bar{\mathbf{x}}, \Delta\bar{\mathbf{x}}) \in \{(\bar{\mathbf{x}}, \Delta\bar{\mathbf{x}}) \in \mathbb{R}^{n_{\bar{\mathbf{x}}}} \times \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} :$$
$$\mathbf{g}_c(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) \leq \mathbf{0}, \tag{3}$$
$$\text{for every } \Delta\bar{\mathbf{x}} \in \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} : \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0}\}\}$$

The robust control problem in Eq. (3) is affected by the design variables, uncertainty associated with design variables, and uncertainty associated with control parameters. Note that the DRC will often yield a more robust solution than the SDRC.

We are ready now to introduce the notion of a decoupled DRC system. A DRC optimization problem is said to be decoupled if the SDRC strategy defined in Eq. (2) and in Eq. (3) yields the solution of the DRC optimization problem in Eq. (1a).

IV. DRC Coupling

We consider now the characterization of the coupling between design and robust control optimization problems. Physically the notion of a decoupled system is an ideal abstraction of reality since few problems are completely decoupled. We are generally more interested in measuring the degree of coupling of the system. In this section, a measure of the degree of coupling between the design and robust control optimization problems is presented. The design and robust control (DRC) optimization problem is first transformed into an equivalent nested optimization problem. Then, the measure of coupling is obtained by comparing the optimality conditions of the nested optimization problem with the SDRC optimization problems.

A. Nested Design and Robust Control Optimization

To characterize coupling, we transform the DRC optimization problem in Eq. (1a) to an equivalent nested (bi-level) optimization problem that guarantees optimality of the DRC problem. In the nested optimization strategy, an outer loop optimizes the system objective with respect to the design, and an inner (nested) loop finds

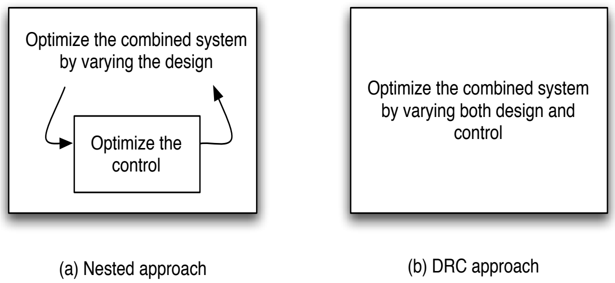

Fig. 1. Comparison Between Nested and DRC Optimization Strategies

the optimal controller for every design generated by the outer loop, as shown in Figure 1. The projected nested optimization (PNO) method introduced by Fathy [10] is used to obtain this nested design and robust control optimization problem.

Let us now apply the projected nested optimization (PNO) method to the design and robust control (DRC) problem in Eq. (1a). To this end, the following sets are defined first:

$\mathbb{X}$ is defined as the set of all designs and controllers for a given uncertainty variable $\Delta\bar{\mathbf{x}}$ that constitute a feasible system (i.e., a system that satisfy the DRC problem constraints) as follows.

$$
\begin{aligned}
\mathbb{X}(\Delta\bar{\mathbf{x}}) = \{(\mathbf{x}_d^T, \mathbf{x}^T(t), \mathbf{u}^T(t))^T \in \mathbb{R}^{n_{\bar{\mathbf{x}}}} : \mathbf{h}_d(\mathbf{x}_d, \mathbf{p}_d) = \mathbf{0},\\
\mathbf{h_c}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) = \mathbf{0},\\
\dot{\mathbf{x}}(t) = \boldsymbol{r}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c),\\
\mathbf{g}_d(\mathbf{x}_d, \mathbf{p}_d) \leq \mathbf{0} \quad , \quad \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0},\\
\bar{\mathbf{x}} \in \{\bar{\mathbf{x}} \in \mathbb{R}^{n_{\bar{\mathbf{x}}}} : \mathbf{g}_c(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) \leq \mathbf{0},\\
\text{for every } \ \Delta\bar{\mathbf{x}} \in \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} : \ \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0}\}\}
\end{aligned}
\tag{4}
$$

$\mathbb{X}_u$ is defined as the set of all uncertainty variables for a given design and control that constitute a feasible system as follows.

$$
\begin{aligned}
\mathbb{X}_u(\mathbf{x}_d, \mathbf{x}(t), \mathbf{u}(t)) = \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} :\\
\mathbf{h_c}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) = \mathbf{0},\\
\dot{\mathbf{x}}(t) = \boldsymbol{r}(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c),\\
\Delta\bar{\mathbf{x}} \in \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} : \mathbf{g}_c(\mathbf{x}(t), \mathbf{u}(t), t, \hat{\mathbf{x}}_d, \hat{\mathbf{p}}_d, \hat{\mathbf{p}}_c) \leq \mathbf{0},\\
\text{for every } \ \Delta\bar{\mathbf{x}} \in \{\Delta\bar{\mathbf{x}} \in \mathbb{R}^{n_{\Delta\bar{\mathbf{x}}}} : \ \mathbf{g}_{\bar{\mathbf{x}}}(\Delta\bar{\mathbf{x}}, \boldsymbol{\alpha}_{\bar{\mathbf{x}}}) \leq \mathbf{0}\}\}
\end{aligned}
\tag{5}
$$

Now let us partition the DRC optimization problem into a design subproblem and a robust control subproblem by partitioning the variable $\bar{\mathbf{x}}$ into a design variable $\mathbf{x}_d$ and a control variable $\mathbf{x}_c$ as follows.

$$\bar{\mathbf{x}} = (\mathbf{x}_d^T, \mathbf{x}_c)^T, \ \ \mathbf{x}_c = (\mathbf{x}^T(t), \mathbf{u}^T(t)) \tag{6}$$

For this given partition, $\mathbb{X}_d$ is defined as the set of all designs for which there exist controllers which constitute feasible systems as follows.

$$\mathbb{X}_d = \{\mathbf{x}_d : \exists \mathbf{x}_c : (\mathbf{x}_d^T, \mathbf{x}_c)^T \in \mathbb{X}\} \tag{7}$$

$\mathbb{X}_c(\mathbf{x}_d)$ is defined as the set of all controllers for a given design that constitute a feasible system as follows.

$$\mathbb{X}_c(\mathbf{x}_d) = \{\mathbf{x}_c : (\mathbf{x}_d^T, \mathbf{x}_c)^T \in \mathbb{X}\} \tag{8}$$

The maximization problem in Eq. (1a) can be represented by the following shorter form

$$F(\mathbf{x}_d, \mathbf{x}(t), \mathbf{u}(t)) = \max_{\Delta\bar{\mathbf{x}} \in \mathbb{X}_u} w_d F_d + w_c \int_{t_i}^{t_f} f_c(.) dt \tag{9}$$

where F is the DRC objective function and $\mathbb{X}_u$ is defined in Eq. (5). Now, the DRC optimization problem in Eq. (1a) can be abstracted as follows.

$$\bar{\mathbf{x}}^* = \underset{\bar{\mathbf{x}} \in \mathbb{X}}{\operatorname{argmin}} \quad F(\bar{\mathbf{x}}) \tag{10}$$

where superscript * denotes system optimality and $\mathbb{X}$ is defined in Eq. (4).

For the partition given in Eq. (6), the following projected nested optimization method [10] has the advantage of guaranteeing system optimality

$$\mathbf{x}_d^* = \arg\min_{\mathbf{x}_d \in \mathbb{X}_d} \ F(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d)) \tag{11}$$

$$\mathbf{x}_c^*(\mathbf{x}_d) = \arg\min_{\mathbf{x}_c \in \mathbb{X}_c(\mathbf{x}_d)} \ F(\mathbf{x}_d, \mathbf{x}_c) \tag{12}$$

the outer loop in Eq. (11) optimizes the DRC objective with respect to the design system, where $\mathbb{X}_d$ is defined in Eq. (7). The inner loop in Eq. (12) searches for the optimal controller for every design produced by the outer loop, where $\mathbb{X}_c(\mathbf{x}_d)$ is defined in Eq. (8). Note that the use of the PNO strategy in Equations (11),(12) requires the determination of the set $\mathbb{X}_d$ of feasible designs prior to their use in the outer loop.

B. Coupling Characterization

Let us consider how the design and robust control (DRC) coupling is found. The DRC coupling can be interpreted as a measure of the degree to which the SDRC solution approaches the DRC solution. Therefore, the importance of the DRC coupling stems from the direct effect it has on the location of the DRC robust system solution. This coupling can be characterized by comparing the optimality conditions of the SDRC problem, Eq. (2) and (3), with the optimality conditions of the nested design and robust control problem, Eq. (11) and Eq. (12).

To this end, assume that the set $\mathbb{X}_d$ is generated by the following constraints

$$\mathbf{h}_d(\mathbf{x}_d, \mathbf{p}_d) = \mathbf{0} \quad , \quad \mathbf{g}_d(\mathbf{x}_d, \mathbf{p}_d) \leq \mathbf{0} \tag{13}$$

therefore, the set $\mathbb{X}_d$ is assumed to be known a priori. Now, consider the SDRC strategy in Eq. (2) and Eq. (3). The design problem and robust control problem can be treated as two separate optimization problems. The Karush-Kuhn-Tucker optimality conditions [17] provide necessary conditions for a point to be optimal.

The Lagrangian L_d for the design optimization problem in Eq. (2) can be stated as follows,

$$L_d = F_d + \boldsymbol{\lambda}_1^T \mathbf{h}_d + \boldsymbol{\mu}_1^T \mathbf{g}_d \tag{14}$$

where $\boldsymbol{\lambda}_1$, $\boldsymbol{\mu}_1$ are the Lagrange multipliers for the equality, inequality constraints, respectively. The first order Karush-Kuhn-Tucker (KKT) stationarity conditions assuming differentiability and regularity [17] are written as

$$
\begin{aligned}
\frac{\partial L_d}{\partial \mathbf{x}_d} = \frac{\partial F_d}{\partial \mathbf{x}_d} + \boldsymbol{\lambda}_1^T \frac{\partial \mathbf{h}_d}{\partial \mathbf{x}_d} + \boldsymbol{\mu}_1^T \frac{\partial \mathbf{g}_d}{\partial \mathbf{x}_d} = \mathbf{0}^T\\
\mathbf{h}_d = \mathbf{0} \quad , \quad \mathbf{g}_d \leq \mathbf{0}\\
\boldsymbol{\lambda}_1 \neq \mathbf{0} \quad , \quad \boldsymbol{\mu}_1 \geq \mathbf{0} \quad , \quad \boldsymbol{\mu}_1^T \mathbf{g}_d = 0
\end{aligned}
\tag{15}
$$

But, the optimality conditions of the SDRC optimal control problems in Eq. (3) and the nested strategy in Eq. (12) are identical by definition. Therefore, we only need to compare the optimality conditions in Eq. (15) with the optimality conditions of the outer nested strategy in Eq. (11).

To derive this outer nested condition, consider the following definition for worst-case control performance

$$f_{max}(\mathbf{x}_d, \mathbf{x}_c) = \max_{\Delta\bar{\mathbf{x}} \in \mathbb{X}_u} \ \int_{t_i}^{t_f} f_c(.) dt \tag{16}$$

the $f_{max}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d))$ can be a discontinuous function, so we utilize the Michel-Penot (M-P) subdifferential [18] to deal with nonsmooth functions. The M-P subdifferential $\partial_x f(x)$ represents a set, but this set becomes a singleton if f is differentiable at x with $\partial_x f(x) = \{\frac{\partial}{\partial x}\}$. When a function $f(x)$ is convex, the M-P subdifferential coincides with the subdifferential in the sense of convex analysis. In addition, the M-P subdifferential set always has a smaller size compared to the Clarke generalized gradient set [19], [18].

Now assume that f_{max} is a Lipschitz function. Moreover, assume that, in terms of the M-P subdifferential, the nondifferentiable Kuhn-Tucker constraint qualification holds [18]. Then the optimality conditions of the outer nested strategy in Eq. (11) can written by utilizing the result by Ye [18] as follows.

$$\begin{gathered} \mathbf{0}^T \in \partial_{\mathbf{x}_d} F_d(\mathbf{x}_d, \mathbf{p}_d) + \boldsymbol{\lambda}_2^T \partial_{\mathbf{x}_d} \mathbf{h}_d(\mathbf{x}_d, \mathbf{p}_d) \\ + \boldsymbol{\mu}_2^T \partial_{\mathbf{x}_d} \mathbf{g}_d(\mathbf{x}_d, \mathbf{p}_d) + \frac{w_c}{w_d} \partial_{\mathbf{x}_d} f_{max}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d)) \\ \mathbf{h}_d = \mathbf{0} \quad , \quad \mathbf{g}_d \leq \mathbf{0} \\ \boldsymbol{\lambda}_2 \neq \mathbf{0} \quad , \quad \boldsymbol{\mu}_2 \geq \mathbf{0} \quad , \quad \boldsymbol{\mu}_2^T \mathbf{g}_d = 0 \end{gathered} \tag{17}$$

where $\partial_{\mathbf{x}_d}$ is the M-P subdifferential with respect to $\mathbf{x}_d$ and $\boldsymbol{\lambda}_2$, $\boldsymbol{\mu}_2$ are the Lagrange multipliers for the equality, inequality constraints, respectively. But since the functions $F_d, \mathbf{h}_d, \mathbf{g}_d$ do not depend on the possibly discontinuous function $f_{max}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d))$, then the M-P subdifferential set is a singleton and the optimality conditions in Eq. (17) can be simplified as follows.

$$\begin{gathered} \mathbf{0}^T \in \frac{\partial F_d}{\partial \mathbf{x}_d} + \boldsymbol{\lambda}_2^T \frac{\partial \mathbf{h}_d}{\partial \mathbf{x}_d} + \boldsymbol{\mu}_2^T \frac{\partial \mathbf{g}_d}{\partial \mathbf{x}_d} + \frac{w_c}{w_d} \partial_{\mathbf{x}_d} f_{max} \\ \mathbf{h}_d = \mathbf{0} \quad , \quad \mathbf{g}_d \leq \mathbf{0} \\ \boldsymbol{\lambda}_2 \neq \mathbf{0} \quad , \quad \boldsymbol{\mu}_2 \geq \mathbf{0} \quad , \quad \boldsymbol{\mu}_2^T \mathbf{g}_d = 0 \end{gathered} \tag{18}$$

Now the optimality conditions in Eq. (15) and Eq. (18) differ by a term Γ_R. This term Γ_R is termed the "DRC coupling" and is defined as follows.

$$\Gamma_R = \frac{w_c}{w_d} \big|\big| \max\{ |\partial_{\mathbf{x}_d} f_{max}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d))| \} \big|\big| \tag{19}$$

where $||.||$ is a vector norm and $|.|$ is the absolute values of each element in the M-P subdifferential. Note that the maximum is optimized over the Michel-Penot subdifferential set to handle discontinuous points. This represents a pessimistic approach because the assumption is always worst-case coupling. Hence, if the coupling Γ_R is zero then the SDRC strategy and the DRC strategy would yield the same robust solution. Now let us consider the case where the worst-case control performance f_{max} is assumed a differentiable function. Then the DRC coupling reduces to

$$\Gamma_R = \frac{w_c}{w_d} \Big|\Big| \frac{\partial f_{max}}{\partial \mathbf{x}_d}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d)) \Big|\Big| \tag{20}$$

Let us now link this coupling defined in Eq. (19) to robustness.

V. Relationship Between Coupling and Robustness

We study now the effect of parameters (i.e., level of uncertainty, objective weights, design and control model parameters) on coupling and robustness of the control system. Varying the parameters establishes a relationship between the DRC coupling and the robustness of the control system. This link between coupling and robustness can be utilized to aid the designer in assessing the parameters' influence on coupling and robustness.

The DRC coupling in Eq. (19) depends on several parameters that were first defined for the DRC optimization problem in Eq. (1a). Let us now study how changing some of these parameters defined in Eq. (1a) affect coupling and robustness. To that end, assume that some of these parameters in Eq. (1a) can be alternatively represented by the parameter $\gamma \in \mathbb{R}$. So the parameter $\gamma \in \mathbb{R}$ can be used to represent the design parameters $\mathbf{p}_d$, control parameters $\mathbf{p}_c$, design weight w_d, control weight w_c, and level of uncertainty $\boldsymbol{\alpha}_{\bar{\mathbf{x}}}$. For example, γ can be used to represent the weights ratio w_c/w_d (i.e. $\gamma \triangleq w_c/w_d$). Hence, the parameters, defined in Eq. (1a), effect on coupling and robustness can be investigated by studying the effect γ on coupling and robustness.

Next, we define our notion of control system robustness emphasizing dependence on the γ parameter.

Definition 1: Consider the following definition of worst-case control performance $f_{max,\gamma}$ emphasizing dependence on the γ parameter

$$f_{max,\gamma}(\bar{\mathbf{x}}, \gamma) = \max_{\Delta\bar{\mathbf{x}} \in \mathbb{X}_u(\gamma)} \int_{t_i}^{t_f} f_c(\mathbf{x}(t,\gamma), \mathbf{u}(t,\gamma), t) dt \tag{21}$$

then control system robustness is defined as the reciprocal $1/f_{max,\gamma}$.

Note that decreasing $f_{max,\gamma}$ increases robustness. Hence, this definition is consistent because increasing $1/f_{max,\gamma}$ increases robustness and vice-versa. Now we examine the relationship between the DRC coupling and the robustness of the control system for two important cases.

The first case occurs when γ is used to represent the level of uncertainty $\boldsymbol{\alpha}_{\bar{\mathbf{x}}}$. In this case, increasing the level of uncertainty implies increased robustness, so studying the relationship between the DRC coupling Γ_R in Eq. (19) and the level of uncertainty $\boldsymbol{\alpha}_{\bar{\mathbf{x}}}$ is representative of the relationship between the DRC coupling Γ_R and the control system robustness $1/f_{max,\gamma}$.

The second case occurs when γ is used to represent the design parameters $\mathbf{p}_d$, control parameters $\mathbf{p}_c$, design weight w_d, and control weight w_c. Both the DRC coupling Γ_R in Eq. (19) and the control system robustness $1/f_{max,\gamma}$ depend on the parameter γ. For different values for this parameter γ, the resulting relationship between the DRC coupling Γ_R in Eq. (19) and the control system robustness $1/f_{max,\gamma}$ can be examined.

A. *Coupling and Robustness Parameter Sensitivity*

Examining the DRC coupling and robustness relationship by varying the parameter γ can be computationally expensive. A more computationally efficient approach is to study the sensitivity of the DRC coupling and robustness with respect to the parameter γ. This sensitivity can be found by taking the gradient of the DRC coupling Γ_R in Eq. (19) and the control system robustness $1/f_{max,\gamma}$ in Eq. (21) with respect to the parameter γ. Assume that the control system robustness $1/f_{max,\gamma}$ is differentiable with respect to γ. Using the chain rule, the sensitivity of the DRC coupling with respect to the control system robustness can be written as follows.

$$\partial_{f_{max,\gamma}^{-1}} \Gamma_R = \partial_\gamma \Gamma_R \frac{\partial \gamma}{\partial f_{max,\gamma}^{-1}} \tag{22}$$

if the M-P subdifferential set $\partial_\gamma \Gamma_R$ is a singleton then

$$\frac{\partial \Gamma_R}{\partial f_{max,\gamma}^{-1}} = \frac{\partial \Gamma_R}{\partial \gamma} \frac{\partial \gamma}{\partial f_{max,\gamma}^{-1}} \tag{23}$$

This expression investigates the sensitivity of the coupling to small changes in robustness. It can be used to estimate the impact on coupling and robustness resulting from a small change in γ.

The coupling sensitivity $\partial_\gamma \Gamma_R$ to the parameter γ in Eq. (23) is found by taking the gradient of Eq. (19). If the coupling in Eq. (19) is assumed differentiable then the following expression can be used to calculate the coupling sensitivity

$$\frac{\partial \Gamma_R}{\partial \gamma} = \frac{w_c}{w_d} \frac{\big|\big| \frac{\partial f_{max}}{\partial \mathbf{x}_d} \big|\big|}{\frac{\partial f_{max}}{\partial \mathbf{x}_d}} \Big(\frac{\partial f_{max}}{\partial \gamma \partial \mathbf{x}_d} + \frac{\partial f_{max}}{\partial \mathbf{x}_c \partial \mathbf{x}_d} \frac{\partial \mathbf{x}_c^*}{\partial \gamma} \tag{24}$$

$$+\Big(\frac{\partial f_{max}}{\partial\gamma\partial\mathbf{x}_c}+\frac{\partial^2 f_{max}}{\partial\mathbf{x}_c^2}\frac{\partial\mathbf{x}_c^*}{\partial\gamma}\Big)\frac{\partial\mathbf{x}_c^*}{\partial\mathbf{x}_d}+\frac{\partial f_{max}}{\partial\mathbf{x}_c}\frac{\partial\mathbf{x}_c^*}{\partial\gamma\partial\mathbf{x}_d}\Big)$$

The coupling sensitivity depends on second order derivatives of the worst-case objective f_{max} with respect to $\mathbf{x}_d$, $\mathbf{x}_c$, and γ. In addition, calculation of the coupling sensitivity requires parametric sensitivity of the control problem solution $\mathbf{x}_c^*$ and the control problem solution sensitivity to $\mathbf{x}_d$ both with respect to the parameter γ.

B. Specific Cases with Known Relationship Between Coupling and Robustness

In the previous section we introduced a sensitivity approach to studying the coupling and robustness relationship. Cases where the parameters appear in a nice, explicit way in the expression for Γ_R will allow us to reduce the computational effort for measuring coupling. An obvious, well known case is the influence of the weights.

Assume that $\gamma \triangleq w_c/w_d$, and that coupling changes are a result of changing the DRC weights w_d, w_c in Eq. (19). Then, as the DRC coupling decreases, robustness of the control system decreases and vice-versa.

Pareto weights and coupling. *Assume that only the DRC weights w_d, w_c in Eq. (1a) can be used to change coupling. Then as the DRC coupling Γ_R decreases, robustness of the control system $1/f_{max}$ decreases and vice-versa.*

This is easily seen by writing Eq. (1a) in the following short form

$$\min_{\bar{\mathbf{x}}\in\ \mathbb{X}}\ \ w_d F_d + w_c f_{max} \tag{25}$$

where the set $\mathbb{X}$ is defined in Eq. (4). Varying the weights w_c/w_d in Eq. (25) correspond to finding the Pareto front. This Pareto front is convex and can be discontinuous because the weighted-sum method may not find points on the non-convex portions of the Pareto front. The case $w_c/w_d \to \infty$ represents maximum control robustness (i.e., minimizing worst-case performance). The case $w_c/w_d \to 0$ represents seeking only the optimal design objective and corresponds to minimum robustness for points on the Pareto curve. Hence, robustness increases as the ratio w_c/w_d increases and decreases as the ratio w_c/w_d decreases. Now let us relate this result to the DRC coupling. From the coupling definition in Eq. (19), increasing the ratio w_c/w_d increases coupling while decreasing the ratio w_c/w_d decreases coupling. Hence, we can conclude that as coupling decreases, robustness of the control system decreases and as coupling increases, robustness of the control system increases.

In the previous case, only the weights w_d, w_c could alter coupling. In the following second case, the weights w_d, w_c restriction is removed, so coupling can be altered by any parameter $\gamma\in\mathbb{R}$ it depends on. The second case does not fully characterize the relationship between coupling and robustness; rather, it studies whether a parameter change causes a decreases or an increase in system robustness. The second case states that as the DRC coupling decreases and tends to zero then decreasing the SDRC control system robustness decreases the DRC control system robustness. Similarly, as the DRC coupling decreases and tends to zero then increasing the DRC control system robustness increases the SDRC control system robustness. To prove this result, we first represent the DRC optimization problem in Eq. (1a) by the following more compact form

$$\bar{\mathbf{x}}_\gamma^* = \arg\min_{\bar{\mathbf{x}}\in\ \mathbb{X}}\ \ w_d f_d + w_c f_{max,\gamma}(\bar{\mathbf{x}},\gamma) \tag{26}$$

The second case can be proved as follows.

Relation of DRC and SDRC computed robustness. *Let $\bar{\mathbf{x}}_\gamma^*$ be the solution of the optimization problem in Eq. (26). Assume that the parameter change from $\gamma=\gamma_1$ to $\gamma=\gamma_2$ decreases the DRC coupling Γ_R, in Eq. (19), making it tend to zero. Then,*

(a) *if the parameter change $\gamma_1 \to \gamma_2$ decreases robustness computed using the SDRC strategy, then robustness computed using the DRC strategy must also decrease;*
(b) *if the parameter change $\gamma_1 \to \gamma_2$ increases robustness computed using the DRC strategy, then robustness computed using the SDRC strategy must also increase.*

(a) Assume that the DRC coupling Γ_R is equal to a non-zero constant T_c, i.e., $\Gamma_R = T_c$. Moreover, assume that the parameter γ is used to alter coupling. Consider utilizing the SDRC optimization strategy to solve Eq. (1a). Let $\bar{\mathbf{x}}_\gamma^s$ represent the SDRC strategy optimal solution resulting from solving Eq. (2) and Eq. (3). The SDRC strategy is guaranteed to find a particular Pareto point that corresponds to the infinite ratio $\frac{w_d}{w_c}\to\infty$ (i.e., design is infinitely more important that control), thus yielding the Pareto point with the maximum worst-control performance. Now, consider the optimization problem in Eq. (26). Then by definition

$$f_{max,\gamma}(\bar{\mathbf{x}}_\gamma^*,\gamma) \le f_{max,\gamma}(\bar{\mathbf{x}}_\gamma^s,\gamma) \quad \bar{\mathbf{x}}_\gamma^*,\bar{\mathbf{x}}_\gamma^s\in\mathbb{X},\gamma\in\mathbb{R} \tag{27}$$

Let $\gamma=\gamma_1$ and suppose that the optimality conditions of the DRC strategy differ from the optimality conditions of the SDRC strategy by a DRC coupling value T_c. Therefore, the SDRC optimum differs from the DRC optimum. As a result, Eq. (27) can be modified as follows.

$$f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^*,\gamma_1) < f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^s,\gamma_1) \quad \bar{\mathbf{x}}_{\gamma_1}^*,\bar{\mathbf{x}}_{\gamma_1}^s\in\mathbb{X} \tag{28}$$

Now let $\gamma=\gamma_2$ and consider the case where the coupling tends to zero, i.e., the problem decouples. In this case, the optimality conditions of the SDRC strategy will approach the DRC optimality conditions as the constant T_c tends to zero. In the limiting case ($T_c=0$) Eq. (27) can be written as follows

$$f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^*,\gamma_2) = f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^s,\gamma_2) \quad \bar{\mathbf{x}}_{\gamma_2}^*,\bar{\mathbf{x}}_{\gamma_2}^s\in\mathbb{X} \tag{29}$$

Now for the parameter change $\gamma_1\to\gamma_2$, assume that the worst-case control performance, computed using the SDRC strategy, has increased

$$f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^s,\gamma_1) < f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^s,\gamma_2) \tag{30}$$

then from Eq. (28), Eq. (29) and Eq. (30)

$$f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^*,\gamma_1) < f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^*,\gamma_2) \tag{31}$$

therefore control system robustness computed using the DRC strategy decreases (i.e., the DRC strategy worst-case control performance increases) as a result of the change from $\gamma_1\to\gamma_2$. Since changing γ_1 to γ_2 decreases coupling, then we can conclude that as robustness computed using the SDRC strategy decreases, then robustness computed using the DRC strategy must also decrease.

(b) Now for the parameter change $\gamma_1\to\gamma_2$, assume that the worst-case control performance computed using the DRC strategy has decreased

$$f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^*,\gamma_2) < f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^*,\gamma_1) \tag{32}$$

then from Eq. (28), Eq. (29) and Eq. (32)

$$f_{max,\gamma_2}(\bar{\mathbf{x}}_{\gamma_2}^s,\gamma_2) < f_{max,\gamma_1}(\bar{\mathbf{x}}_{\gamma_1}^s,\gamma_1) \tag{33}$$

therefore control system robustness computed using the SDRC strategy increases (i.e., the SDRC strategy worst-case control performance decreases) as a result of the change from γ_1 to γ_2. Since changing γ_1 to γ_2 decreases coupling, then we can conclude that as robustness computed using the DRC strategy increases, then robustness computed using the SDRC strategy must also increase.

Now let us illustrate the results of this section with an example.

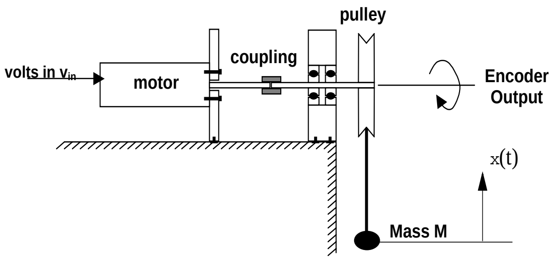

Fig. 2. Elevator Setup

VI. Elevator Control Example

The following example is a simple elevator control problem which is sufficient to illustrate the key ideas of the DRC coupling and robustness relationship. The example focuses on demonstrating the properties of the DRC coupling presented in the previous section.

Consider a simple elevator model consisting of an electric motor receiving an input voltage v_i and driving a pulley that lifts a mass M using a cable [10]. The mass M starts from rest at zero elevation and is displaced from its initial position $x(t)$ as shown in Figure 2. If the motor's torque and back electromotive force constants are assumed to be equal and denoted by k_t, and if the motor's armature inductance, bearing friction, cable mass and compliance, and the pulley and shaft inertias are neglected, then the motion of the elevator is governed by the following equation

$$M\ddot{x}(t) + \frac{k_t^2}{r^2 R_a}\dot{x}(t) = \frac{k_t}{rR_a}v_i - Mg \tag{34}$$

where R_a is the armature resistance, r is the pulley radius, and g is the acceleration of gravity. Now suppose that the voltage v_i is given by

$$v_i = \frac{rR_aMg}{k_t} + k_p(x_{ref} - x(t)) \tag{35}$$

where k_p is a proportional feedback gain and x_{ref} is a reference elevation. The elevator control problem objective is to minimize the weighted sum of an approximation of the system rise time with the reciprocal of mass that can be lifted at this speed. The problem constraints guarantee stability, maximum allowable motor input voltage and minimum and maximum allowable payloads. In addition, we constrain the damping ratio to give an upper limit on overshoot; furthermore, the rise time approximation used is considered valid in this range. Hence, the elevator optimization problem can be stated as follows.

$$\begin{aligned} &\min_{M,k_p} \left(w_d\frac{1}{M} + w_c 1.8\sqrt{\frac{rR_aM}{k_tk_p}} \right) \\ &\text{subject to} \\ &k_p \geq 0\ ,\ \frac{rR_aMg}{k_t} + k_p x_{ref} \leq V_{max} \\ &M_{min} \leq M \leq M_{max}\ ,\ \frac{k_t^{1.5}}{2r^{1.5}}\frac{1}{\sqrt{R_aMk_p}} \geq \zeta_{min} \end{aligned} \tag{36}$$

Now assume that the mass M varies due to random occupancy of the elevator by passengers. This randomness is assumed to be represented by a 50% set-based percentage uncertainty. For the parameter values $x_{ref} = 10\ cm$, $V_{max} = 50\ volts$, $M_{min} = 0.1\ kg$, $M_{max} = 7.97\ kg$, $w_d = 0.1, w_c = 0.9$, and $\zeta_{min} = 0.7$, the elevator combined design and robust control optimization problem can be represented in terms of the DRC optimization problem in Eq. (1a) as follows.

$$\begin{aligned} &\min_{M,k_p}\max_{\rho} \left(0.1\frac{1}{M} + 0.9(1.8)\sqrt{\frac{rR_a\hat{M}}{k_tk_p}} \right) \\ &\text{subject to} \\ &k_p \geq 0\ ,\ 1-\alpha \leq \rho \leq 1+\alpha\ ,\ 0.1 \leq M \leq 7.97 \\ &(M,k_p,\rho) \in \{(M,k_p,\rho) : k_p \leq 500 - 98.1\frac{rR_a\hat{M}}{k_t}, \\ &\frac{k_t^{1.5}}{2r^{1.5}}\frac{1}{\sqrt{R_a\hat{M}k_p}} \geq 0.7, \\ &\text{for every } \rho \in \{\rho :\ 1-\alpha \leq \rho \leq 1+\alpha\}\} \\ &\text{where} \quad \hat{M} = \rho M \end{aligned} \tag{37}$$

In Eq. (1a) notation, $\mathbf{x}_d = M$, $\mathbf{x}_c = k_p$, $\bar{\mathbf{x}} = (M, k_p)^T$, $\Delta\mathbf{x}_d = \rho$, and $\boldsymbol{\alpha}_{\bar{\mathbf{x}}} = \alpha = 0.5$. Note that feasibility robustness is maintained since for all values of uncertainty the constraints are always satisfied. Let us now solve this minimax problem using the SDRC optimization strategy. The SDRC partitions the system optimization problem into a design and a robust control optimization problems. Here, the design objective is to maximize the elevator payload and the robust control objective is to minimize the worst-case rise time as follows.

$$\text{Design problem: } M^s = \underset{M}{\text{argmin}}\ \frac{1}{M} \quad \text{s.t. } 0.1 \leq M \leq 7.97$$

$$\begin{aligned} &\text{Control problem: } k_p^s = \underset{k_p}{\text{argmin}}\max_{\rho}\ 1.8\sqrt{\frac{rR_a\hat{M}}{k_tk_p}} \\ &\text{subject to} \\ &k_p \geq 0\ ,\ 1-\alpha \leq \rho \leq 1+\alpha \\ &(M,k_p,\rho) \in \{(M,k_p,\rho) : k_p \leq 500 - 98.1\frac{rR_a\hat{M}}{k_t}, \\ &\frac{k_t^{1.5}}{2r^{1.5}}\frac{1}{\sqrt{R_a\hat{M}k_p}} \geq 0.7, \\ &\text{for every } \rho \in \{\rho :\ 1-\alpha \leq \rho \leq 1+\alpha\}\} \\ &\text{where} \quad \hat{M} = \rho M \end{aligned} \tag{38}$$

The solution to these optimization problems for $r/k_t = 0.0133, R_a = 1$ can found to be $M^* = 3.9946$, $k_p^* = 189.53$, $\rho^* = 1.5$, and $f^*_{max} = 0.037$ for the DRC problem in Eq. (37), and $M^s = 7.97$, $k_p^s = 134.13$, $\rho^s = 1.5$, and $f^s_{max} = 0.0621$ for the SDRC problem in Eq. (38).

1) Relationship between coupling and robustness by varying the level of uncertainty: Let us now study the effect of varying the level of uncertainty α on coupling and control system robustness. Let the parameter γ be defined as the level of uncertainty α (i.e., $\gamma \triangleq \alpha$). Figure 3 shows the relationship between coupling calculated using Eq. (19) and the level of uncertainty α. The coupling exhibits a monotonically increasing relationship with the level of uncertainty α, so demanding more robustness (i.e., robustness to a higher level of uncertainty α) increases coupling between design and robust control. Therefore, the SDRC should be solved here only for small levels of uncertainty, while the DRC should be solved for high levels of uncertainty. This result can also be obtained in a more computationally efficient manner by utilizing the coupling sensitivity with respect to the parameter γ. For 10% uncertainty, the coupling sensitivity is calculated using Eq. (24) to yield $d\Gamma_R/d\gamma = 0.0274$ at $\alpha = 0.1$. The coupling sensitivity is positive, so increasing the level of uncertainty increases coupling locally. Hence, the designer can use local information to predict the coupling and robustness relationship.

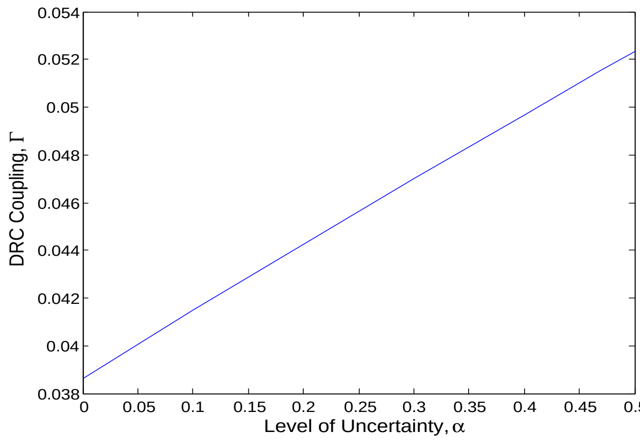

Fig. 3. Relationship Between Coupling and Level of Uncertainty α

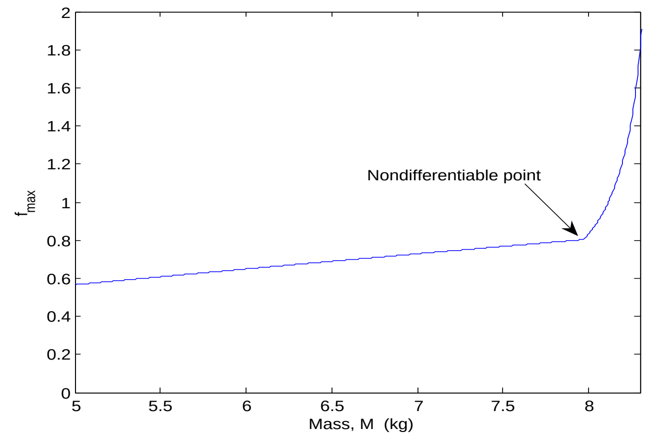

Fig. 5. Illustration of the nondifferentiability associated with calculating the coupling

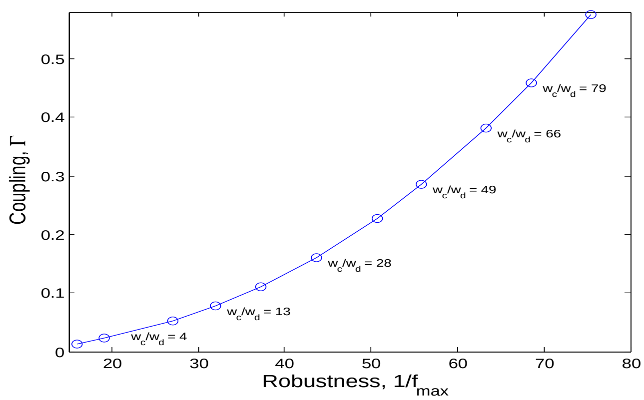

Fig. 4. Relationship Between Coupling and Robustness as Result of Changing the Weights w_d, w_c

2) Relationship between coupling and robustness by varying the Pareto weights: We can also illustrate the first case in Section V.B. The observation states that if the DRC weights w_d, w_c are used to change coupling, then as the DRC coupling Γ_R decreases, robustness of the control system decreases. Figure 4 shows the relationship between coupling and robustness that results from changing the DRC weights w_d, w_c. The relationship in the figure illustrates the trade-off between coupling and robustness. Then from Figure 4, decreasing the DRC coupling Γ_R decreases robustness of the control system.

3) Relationship between coupling and robustness by varying the ratio rR_a/k_t: Let us now illustrate the second case of Section V.B. Consider studying the effect of the ratio rR_a/k_t changes on coupling and robustness. To that end, set $w_d = 0.1, w_c = 0.9$ and assume that coupling can only be altered by changing the ratio $\gamma \triangleq rR_a/k_t$, where γ was introduced in Eq. (21).

Now let us investigate the effect of changing the pulley radius from $\gamma_1 = rR_a/k_t = 0.405$ $(r/k_t = 0.013, R_a = 30.4)$ to $\gamma_2 = rR_a/k_t = 0.1389$ $(r/k_t = 0.1389, R_a = 1)$. For $\gamma_1 = 0.405$, coupling must be calculated using Eq. (19) because f_{max} is nondifferentiable, as shown in Figure 5. At the SDRC solution $M^s = 7.97$, the M-P subdifferential set is $\partial_{\mathbf{x}_d} f_{max} = [0.68, 9.35]$. The coupling is found by maximizing the following M-P subdifferential set $\max[0.68, 9.35]$ to yield $\Gamma_{R1} = \frac{w_c}{w_d} \left|\left| \max\{|\partial_{\mathbf{x}_d} f_{max}(\mathbf{x}_d, \mathbf{x}_c^*(\mathbf{x}_d))|\}\right|\right| = 9.35$. For $\gamma_2 = 0.1389$, the M-P subdifferential set is singleton and the coupling is $\Gamma_{R2} = 9.35$. Table I(a) illustrates the effect of changing from γ_1 to γ_2 by solving the DRC optimization problem in Eq. (37) for $\gamma_1 = 0.405$ and $\gamma_2 = 0.1389$. This solution is then compared to solving the optimization problem in Eq. (38) using the SDRC strategy.

The solution of the SDRC problem in Eq. (38) is $f^s_{max,\gamma_1} = 0.804$ for γ_1 and $f^s_{max,\gamma_2} = 1.16$ for γ_2, so changing from γ_1 to γ_2 decreased robustness of the SDRC problems (i.e. $f^s_{max,\gamma_1} < f^s_{max,\gamma_2}$) . In addition, coupling decreased from $\Gamma_{R1} = 9.35$ for γ_1 to $\Gamma_{R2} = 0.68$ for γ_2. Now the observation in Section V.B states that the robustness of the control system must decrease. Indeed, this can be confirmed by looking at the DRC solution for Eq. (37) which is $f^s_{max,\gamma_1} = 0.16$ for γ_1 and $f^s_{max,\gamma_2} = 0.1975$ for γ_2, so changing from γ_1 to γ_2 decreased robustness of the DRC problems (i.e. $f^*_{max,\gamma_1} < f^*_{max,\gamma_2}$). Therefore, decreased coupling and decreased robustness of the control system obtained using the SDRC strategy resulted in decreasing the robustness of the control system obtained using the DRC strategy.

Let us study the effect of changing the ratio rR_a/k_t from $\gamma_1 = rR_a/k_t = 0.405$ $(r/k_t = 0.013, R_a = 30.4)$ to $\gamma_2 = rR_a/k_t = 0.0133$ $(r/k_t = 0.0133, R_a = 1)$. For $\gamma_1 = 0.405$, the coupling is found by maximizing the following M-P subdifferential set $[0.68, 9.35]$ to yield $\Gamma_{R1} = 9.35$. For $\gamma_2 = 0.0133$, the M-P subdifferential set is singleton and the coupling is $\Gamma_{R2} = 0.0525$. Table I(b) illustrates the effect of changing from γ_1 to γ_2 by solving the DRC optimization problem in Eq. (37) for $\gamma_1 = 0.405$ and $\gamma_2 = 0.0133$. This solution is then compared to solving the optimization problem in Eq. (38) using the SDRC strategy.

The solution of the DRC problem in Eq. (37) is $f^*_{max,\gamma_1} = 0.16$ for γ_1 and $f^*_{max,\gamma_2} = 0.037$ for γ_2, so changing from γ_1 to γ_2 increased robustness of the DRC problems (i.e. $f^*_{max,\gamma_2} < f^*_{max,\gamma_1}$). In addition, coupling decreased from $\Gamma_{R1} = 9.35$ for γ_1 to $\Gamma_{R2} = 0.0525$ for γ_2. Our earlier observation stated that the robustness of the control system must increase. Indeed, this can be confirmed by looking at the SDRC solution for Eq. (38) which is $f^s_{max,\gamma_1} = 0.804$ for γ_1 and $f^s_{max,\gamma_2} = 0.0621$ for γ_2, so changing from γ_1 to γ_2 increased robustness of the SDRC problems (i.e., $f^s_{max,\gamma_2} < f^s_{max,\gamma_1}$). Therefore, decreased coupling and increased robustness of the control system obtained using the DRC strategy resulted in increasing the robustness of the control system obtained

γ	SDRC				DRC				Coupling
	M^s	k_p^s	ρ^s	f_{max}^s	M^*	k_p^*	ρ^*	f_{max}^*	
$\gamma_1 = 0.405$	7.97	24.27	1.5	0.804	0.9274	71.341	1.5	0.16	$\Gamma_{R1} = 9.35$
$\gamma_2 = 0.1389$	7.97	3.99	1.5	1.16	0.7513	12.999	1.5	0.1975	$\Gamma_{R2} = 0.68$

(a)

γ	SDRC				DRC				Coupling
	M^s	k_p^s	ρ^s	f_{max}^s	M^*	k_p^*	ρ^*	f_{max}^*	
$\gamma_1 = 0.405$	7.97	24.27	1.5	0.804	0.9274	71.341	1.5	0.16	$\Gamma_{R1} = 9.35$
$\gamma_2 = 0.0133$	7.97	134.13	1.5	0.0621	3.9946	189.53	1.5	0.037	$\Gamma_{R2} = 0.0525$

(b)

TABLE I

COMPARISON OF DRC AND SDRC ROBUSTNESS SOLUTIONS WITH RESPECT TO PARAMETER CHANGES IN COUPLING TERM: (A) DECREASED ROBUSTNESS (B) INCREASED ROBUSTNESS

using the SDRC strategy.

VII. CONCLUSION

This article introduced a coupling strength measure for the design and robust control optimization problems. The coupling strength measure was found by comparing the optimality conditions of the design and robust control (DRC) optimization problem with the sequential design and robust control (SDRC) optimization problems.

By varying the parameters, we studied the resulting relationship between the DRC coupling and the robustness of the control system. Designers could use this information to assess the parameters' influence on coupling and robustness. In addition, we presented a more computationally efficient approach to examining this coupling and robustness relationship by studying the sensitivity of the DRC coupling and robustness with respect to these parameters.

Future work should consider optimizing coupling to improve robustness as opposed to solving the DRC optimization problem. The properties of coupling proved in this article can help with the formulation of this coupling optimization problem. Future work should also consider investigating the coupling relationship to uncertainty for fuzzy set and probabilistic approaches.

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