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An SLP Filter Algorithm for Probabilistic Analytical Target Cascading

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1 Abstract

Decision-making under uncertainty is particularly challenging in the case of multidisciplinary, multilevel system optimization problems. Subsystem interactions cause strong couplings, which may be amplified by uncertainty. Thus, effective coordination strategies can be particularly beneficial. Analytical target cascading (ATC) is a deterministic optimization method for multilevel hierarchical systems, which was recently extended to probabilistic formulations in the so-called probabilistic design. Solving the optimization problem requires propagation of uncertainty, namely, evaluating or estimating the output distributions. This uncertainty propagation can be a very challenging and computationally expensive task for highly nonlinear functions. In order to overcome the general difficulty in uncertainty propagation, this article extends the use of the SLP algorithm to the probabilistic ATC formulation. By linearizing and solving a problem successively, the algorithm takes advantage of the simplicity and ease of uncertainty propagation for a linear system. A suspension strategy, developed for a deterministic SLP-based ATC strategy, is applied to reduce computational cost by suspending the analyses of subsystems that do not need considerable redesigns. The accuracy and effectiveness of the proposed SLP-based PATC strategy is demonstrated with several numerical examples.

2 Keywords: probabilistic analytical target cascading, sequential linear programming, complex system design under uncertainty, suspension strategy

3 Introduction

Engineers make decisions under various forms of uncertainty. Moreover, many products today are large and complex systems whose design requires multidisciplinary analyses involving significant interactions. Inclusion of uncertain quantities in these interactions can strongly couple subsystems to each other [1]. Use of decomposition strategies in multidisciplinary design optimization (MDO) of such system may be the only available solution approach.

Compared to deterministic MDO, little research has been conducted to address uncertainty in multidisciplinary or multi-level system design optimization problems. Batill et al. [2] discuss the challenges faced for including uncertainty in MDO. In Du and Chen [3], system uncertainty analysis (SUA) and concurrent subsystem uncertainty analysis (CSSUA) are proposed to improve the efficiency in finding the propagation of uncertainty within one coupled multidisciplinary problem. Collaborative reliability analysis is proposed also by Du and Chen to improve the efficiency of reliability evaluation in MDO under uncertainty [4]. In this work, a single loop procedure is used to satisfy the interdisciplinary consistency in reliability analysis. Thus, past research associated with MDO under uncertainty has focused on how to define qualitatively uncertainty within MDO, while little attention has been paid on the hierarchical system design aspect of the problem.

Analytical target cascading (ATC) is an optimization method for multilevel hierarchical systems typically partitioned into physical subsystems or objects (see Figure 1.a) [5]. Each block in the hierarchical structure, referred to as an element, is an optimization sub-problem. An element can be coupled with only one parent element but with multiple children elements. The linking variables between a parent and children are design targets and analysis responses. Targets are set by parents and propagated to their children; the children are optimized to obtain responses that are as close to the targets as possible. Thus, targets and responses are updated and coordinated iteratively to achieve consistent values globally.

Deterministic ATC was recently extended to probabilistic formulations, the so-called probabilistic analytical target cascading (PATC), by Kokkolaras et al. [7] using mean values to represent random linking variables. Liu et al. [8] generalized the formulation with general probabilistic characteristics. In PATC, we consider a

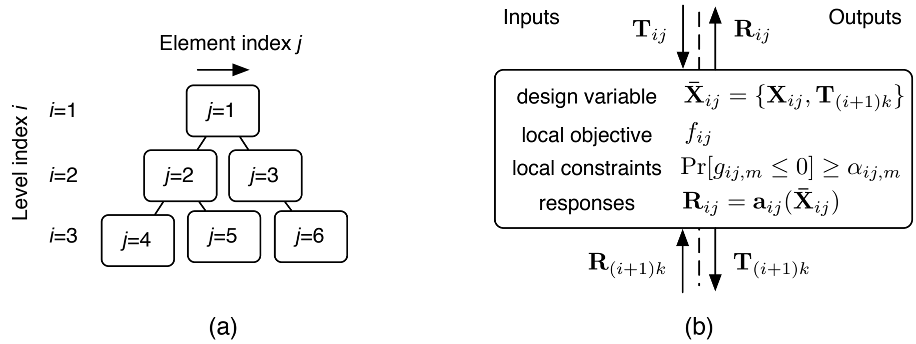

Figure 1: Example of index notation in ATC and information flow for an element O_{ij} [6]

probabilistic all-in-one (PAIO) system design problem expressed as

$$\begin{array}{rl} \min\limits_{\mathbf{X}} & \mathrm{E}[f(\mathbf{X})] \\ \text{subject to} & \Pr[g_m(\mathbf{X}) \leq 0] \geq \alpha_m, \quad m = 1, ..., M_c, \end{array} \tag{1}$$

where M_c is the number of constraints. In Eq.(1), f and $\mathbf{X}$ are the system objective function and the vector of all random design variables, respectively. Design constraints are expressed in an probabilistic feasibility formulation, in which the probability of satisfying $g_m(\mathbf{X}) \leq 0$ is greater than the required reliability level α_m. For target matching problems, $f(\mathbf{X})$ is expressed as $||\mathbf{T} - \mathbf{R}(\mathbf{X})||$ where $\mathbf{T}$ and $\mathbf{R}$ are the system's targets and responses. Note that no probabilistic equality constraints are included because satisfying such constraints under uncertainty is not meaningful.

Assuming that the system objective and constraints are separable, Eq.(1) is decomposed hierarchically into N elements at M levels. Quantities with indices ij are related to element j at level i. As shown in Figure 1.b, $\mathbf{X}_{ij}$, $\mathbf{T}_{ij}$ and $\mathbf{R}_{ij}$ denote local design variables, targets and responses to the element O_{ij} while f_{ij}, $\mathbf{g}_{ij}$ and $\mathbf{a}_{ij}$ are local objective, constraint and response functions. Consistency between elements is relaxed and coordinated through penalty functions. Then the generalized PATC formulation for an element O_{ij} with a quadratic penalty function is :

$$\begin{array}{rl} \text{Given} & \mathbf{T}_{ij}, \mathbf{R}_{(i+1)k}, \\ \min\limits_{\bar{\mathbf{X}}_{ij}} & \mathrm{E}[f_{ij}(\bar{\mathbf{X}}_{ij})] + ||\mathbf{w}_{ij} \circ (\mathbf{T}_{ij} - \mathbf{R}_{ij})||_2^2 + \sum\limits_{k \in \mathcal{C}_{ij}} ||\mathbf{w}_{(i+1)k} \circ (\mathbf{T}_{(i+1)k} - \mathbf{R}_{(i+1)k})||_2^2 \\ \text{subject to} & \Pr[g_{ij,m}(\bar{\mathbf{X}}_{ij}) \leq 0] \geq \alpha_{ij,m}, \quad m = 1, ..., M_{c,ij} \\ \text{where} & \mathbf{R}_{ij} = \mathbf{a}_{ij}(\bar{\mathbf{X}}_{ij}), \quad \bar{\mathbf{X}}_{ij} = [\mathbf{X}_{ij}, \mathbf{T}_{(i+1)k}], \quad \forall k \in \mathcal{C}_{ij}, \quad \forall j \in \mathcal{E}_i, i = 1, ..., N, \end{array} \tag{2}$$

where $\mathcal{C}_{ij}$ is the set of the children of element j at level i and $\mathcal{E}_i$ is the set of elements at level i. In Eq.(2), the $\circ$ operation indicates the component-wise multiplication of two vectors such that $\{a_1, ..., a_k\}^T \circ \{b_1, ..., b_k\}^T = \{a_1 b_1, ..., a_k b_k\}^T$. For detailed ATC and PATC formulations, readers are referred to [6, 7, 8, 9, 10].

As pointed out in Liu et al. [8], the choice of random variable representation is an important issue in MDO under uncertainty. A popular way to define uncertainty is using random variables, assuming that their probability density functions (PDFs) can be inferred. These distributions are assumed as inputs to the optimization problem. Solving the optimization problem requires propagation of this uncertainty, namely, evaluating or estimating the output distributions associated with the optimized values. This uncertainty propagation can be a very challenging and computationally expensive task for highly nonlinear functions. Moreover, the output distributions of response functions are typically non-normal, which might be considerably difficult to infer. Thus, in the previously published PATC formulations, the first few moments are used as targets and responses to be matched even though computing the solution is still very expensive.

In order to overcome the general difficulty in uncertainty propagation, Chan et al. [11] introduced sequential linear programming (SLP) to solve probabilistic design optimization for a single system, with the goal of achieving an appropriate balance between accuracy, efficiency and convergence behavior. By linearizing and solving a problem successively, the algorithm takes advantage of the simplicity and ease of uncertainty propagation for a linear system. The SLP algorithm employs a filtering and trust region strategy to prove global convergence of deterministic and probabilistic optimization for a single system [11, 12, 13]. In a more recent study, a SLP algorithm was employed as an alternate coordination strategy to solve deterministic ATC problems [14]. Several examples were used to demonstrate that SLP-based coordination can successfully reduce the number of function evaluations without losing solution accuracy.

In this article, we employ SLP as an alternative coordination strategy to solve PATC problems. In the proposed algorithm, probabilistic constraints are approximated by equivalent deterministic linear constraints and the uncertainty propagation in the linearized subproblem is obtained easily. The linking variables are represented only with means and standard deviations, and so consistency of random variables does not require significant computation in estimating and matching distributions. Further a subsystem suspension strategy, developed for specially an SLP-based ATC, is used to reduce computational cost by suspending the analyses of subsystems that do not need considerable redesign, based on the size of trust regions and the step size of target values. The effectiveness of the proposed SLP-based PATC strategy is demonstrated by comparing results for several examples to those obtained from previously proposed solution strategies.

The article is organized as follows. In Section 2, SLP-based PATC is formulated and the method of updating standard deviations of linking variables is discussed. Section 3 explains briefly the suspension strategy for SLP-based PATC. Illustrative test examples are presented in Section 4, followed by conclusions in Section 5.

4 SLP-based probabilistic analytical target cascading formulations

In SLP-based PATC, a Probabilistic Linearized ATC (PLATC) problem is created from a nonlinear PATC problem and is solved using the "standard" ATC strategy. By solving PLATC successively, the algorithm converges to a solution of the original nonlinear PATC problem. Similar to [11], probabilistic constraints are approximated with equivalent deterministic constraints by either the first or second order reliability method (FORM/SORM). In FORM/SORM [15, 16] standard deviations must be known. Therefore, the means of linking variables in this article are treated as optimization variables, while their standard deviations are estimated at every iteration. The details of PLATC formulation are explained in Section 4.1 while Section 4.2 includes a review on how the standard deviations were handled in the previous PATC literature and a discussion on the updating method for linking variables.

4.1 PLATC subproblem formulation

In Chan et al. [11], an LP subproblem is constructed from PAIO using either FORM or SORM. Both methods approximate the constraints in linear or quadratic functions at the most probable point (MPP), whose distance to the origin in the standard normal space is minimal. The probability of failure is calculated for the approximated function. Even though FORM is simple and less expensive, the linear approximation may cause large errors for highly nonlinear constraints [17, 18]. Thus, in [11], FORM and SORM are judiciously applied to constraints based on the following criteria.

For a constraint $g'_m \equiv \alpha_m - \Pr[g_m(\mathbf{X}) \leq 0] \leq 0$,

$$\text{if } g'_m < -\delta \quad \text{or} \quad E_m = \Phi(-\beta_m)\left\|\prod_p (1+\beta_m\kappa_p)^{-1/2} - 1\right\| \leq E^a, \quad \text{FORM is applied,} \tag{3}$$

$$\text{if } g'_m \geq -\delta \quad \text{and} \quad E_m = \Phi(-\beta_m)\left\|\prod_p (1+\beta_m\kappa_p)^{-1/2} - 1\right\| > E^a, \quad \text{SORM is applied,} \tag{4}$$

where δ is a small positive number that allows a buffer to g_m and Φ is the standard normal cumulative distribution function. In these criteria, E_m indicates an error between probability of failure estimated by FORM and SORM, and E^a is the tolerance of the error. Also, β_m and κ_p denote the reliability index and the pth principal curvature of g_m at MPP. The principal curvatures $\boldsymbol{\kappa}$ of g_m are calculated as the eigenvalues of the matrix $\mathbf{A}$, given by

$$\mathbf{A} = \frac{\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}}{||\nabla g_m(\mathbf{x}_{\mathrm{M}})||} \tag{5}$$

where $\mathbf{x}_{\mathrm{M}}$ is the MPP of g_m, $\mathbf{D}$ is the Hessian of the g_m at $\mathbf{x}_{\mathrm{M}}$, and $\mathbf{B}$ is a matrix orthogonal to $\mathbf{B_0}$, given by

$$\mathbf{B_0} = \begin{bmatrix} I_{(n-1)\times(n-1)} & & 0 \\ \frac{\partial g_m}{\partial x_1} \; \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}. \tag{6}$$

In Eq.(4), the first criterion indicates the activity of the constraint, so called δ-activity, while the second criterion takes into account the curvature of g_m. Since δ-activity is easy to check, the curvature criterion is applied only if the constraint is δ-active.

In Han and Papalambros [14], an SLP coordination algorithm is applied for ATC problems. In the paper, weighted infinity norms are introduced for penalty functions in order to maintain the linearity of subproblem formulations. Thus, instead of quadratic penalty function terms in the objective, the maximum deviations of consistency constraints, ϵ_{ij}, are used, expressed as

$$\min \quad ||\mathbf{w}_{ij} \circ (\boldsymbol{\mu}_{\mathbf{T}ij} - \boldsymbol{\mu}_{\mathbf{R}ij})||_2^2 \quad \Rightarrow \quad \begin{cases} \quad \min & \epsilon_{ij} \\ \text{subject to} & -\boldsymbol{\epsilon}_{ij} \leq \mathbf{w}_{ij} \circ (\boldsymbol{\mu}_{\mathrm{T}_{ij}} - \boldsymbol{\mu}_{\mathrm{R}_{ij}}) \leq \boldsymbol{\epsilon}_{ij}, \end{cases} \tag{7}$$

where $\boldsymbol{\epsilon}_{ij}$ is a column vector whose components are equal to ϵ_{ij}. Applying constraints approximated by FORM/SORM and infinity norms, the equivalent PLATC subproblem at iteration l with trust region radius $\rho^l > 0$ is given by

$$\begin{array}{rl}
\min & \nabla f_{ij}^l(\boldsymbol{\mu}_{\bar{\mathbf{x}}_{ij}}^l)^T \bar{\mathbf{d}}_{ij}^l + \epsilon_{ij} + \sum\limits_{k \in \mathcal{C}_{ij}} \epsilon_{(i+1)k} \\
\text{with respect to} & \bar{\mathbf{d}}_{ij}^l, \epsilon_{ij}, \epsilon_{(i+1)1}, ..., \epsilon_{(i+1)n_{ij}} \\
\text{subject to} & \nabla g_{ij,m}^l(\bar{\mathbf{x}}_{\mathrm{M}_{ij,m}}^l)^T \bar{\mathbf{d}}_{ij}^l + g_{ij,m}^l(\bar{\mathbf{x}}_{\mathrm{M}_{ij,m}}^l) \leq \mathbf{0} \qquad \text{if } g_{ij,m} \text{ satisfies Eq.(3)}, \\
& \nabla g_{ij,m}^l(\bar{\mathbf{x}}_{\mathrm{S}_{ij,m}}^l)^T \bar{\mathbf{d}}_{ij}^l + g_{ij,m}^l(\bar{\mathbf{x}}_{\mathrm{S}_{ij,m}}^l) \leq \mathbf{0} \qquad \text{if } g_{ij,m} \text{ satisfies Eq.(4)}, \\
& -\boldsymbol{\epsilon}_{ij} \leq \mathbf{w}_{ij} \circ (\boldsymbol{\mu}_{\mathbf{T}_{ij}} + \mathbf{d}_{\boldsymbol{\mu}_{\mathbf{T}_{\mathrm{ij}}}} - \boldsymbol{\mu}_{\mathbf{R}_{ij}} - \mathbf{d}_{\boldsymbol{\mu}_{\mathbf{R}_{\mathrm{ij}}}}) \leq \boldsymbol{\epsilon}_{ij}, \\
& -\boldsymbol{\epsilon}_{(i+1)k} \leq \{\mathbf{w} \circ (\boldsymbol{\mu}_{\mathbf{T}} + \boldsymbol{\mu}_{\mathbf{d_T}} - \boldsymbol{\mu}_{\mathbf{R}} - \boldsymbol{\mu}_{\mathbf{d_R}})\}_{(i+1)k} \leq \boldsymbol{\epsilon}_{(i+1)k}, \\
& ||\bar{\mathbf{d}}_{ij}^l||_\infty \leq \rho^l, \\
\text{where} & \bar{\mathbf{x}}_{\mathrm{M}_{ij,m}}^l = \boldsymbol{\mu}_{\bar{\mathbf{x}}_{ij}}^l + \boldsymbol{\sigma}_{\bar{\mathbf{x}}_{ij}}^l \beta_{t,m} \dfrac{\nabla g_{ij,m}^l}{||\nabla g_{ij,m}^l||}, \quad \bar{\mathbf{x}}_{\mathrm{S}_{ij,m}}^l = \boldsymbol{\mu}_{\bar{\mathbf{x}}_{ij}}^l + \boldsymbol{\sigma}_{\bar{\mathbf{x}}_{ij}}^l \beta_{\mathrm{S},m}^l \dfrac{\nabla g_{ij,m}^l}{||\nabla g_{ij,m}^l||} \\
& \boldsymbol{\mu}_{\mathbf{R}_{ij}}^l = \mathbf{a}_{ij}^l(\boldsymbol{\mu}_{\bar{\mathbf{x}}_{ij}}^l) + \nabla \mathbf{a}_{ij}^l(\boldsymbol{\mu}_{\bar{\mathbf{x}}_{ij}}^l)^T \bar{\mathbf{d}}_{ij}^l \\
& \bar{\mathbf{d}}_{ij}^l = [\mathbf{d}_{\boldsymbol{\mu}_{\mathbf{X}_{ij}}}^l, \mathbf{d}_{\boldsymbol{\mu}_{\mathbf{T}_{(i+1)k}}}^l], \quad \forall k \in \mathcal{C}_{ij}, \quad \forall j \in \mathcal{E}_i, \quad i = 1, ..., N,
\end{array} \tag{8}$$

where $\beta_{t,m}$ is the target reliability index for $g_{ij,m}$ while $\beta_{\mathrm{S},m}^l$ is obtained by solving

$$\Phi(-\beta_{\mathrm{S},m}^l) \prod_p (1 + \beta_{\mathrm{S},m}^l \kappa_p^l)^{-1/2} - (1 - \alpha_m) = 0. \tag{9}$$

Chan et al. [11] showed that the convergence proof of the SLP-filter algorithm can be extended for problems with probabilistic constraints. Similarly, we can replicate the convergence arguments for LATC presented in [14], which shows that the convergence proof of the SLP-filter algorithm can be extended to ATC formulations with consistency constraints under Mangasarian-Fromowitz constraint qualification [19].

4.2 Standard deviation of linking variables

The linking variables can include shared design variables, coupling variables or both. Coupling variables are analysis outputs of one element that are inputs to its parent while shared design variables are design variables that are inputs to multiple elements. If linking variables are design variables and their standard deviations are known in the original PAIO problem, their standard deviations are also constant parameters in PATC. The distributions of the other random linking variables, however, are dependent on the design variables of elements and likely to be non-normal. As pointed out in the previous PATC publications, matching the whole distribution is impractical because the computational cost of coordination increases substantially with the dimension of linking variables. In order to address the issue, Kokkolaras et al. [7] matched the mean values of responses for consistency while their PDFs were estimated from its children elements using an technique base on advanced mean value (AMV) method [20]. On the other hand, Liu et al. [8] had the first two moments matched. Based on their generalized formulations, higher moments can be included for higher accuracy even though matching higher order moments increases the dimension of linking variables. In order to solve PATC with respect to the standard deviations of random variables, Monte Carlo simulation (MCS) with 100,000 samples is applied, which makes the algorithm computationally expensive.

In this paper, a scheme similar to [7] is used for standard deviations. Since PLATC consists of linear functions, the resulting distributions of linking variables are normal, if the distributions of design variables are assumed to be normal. Thus, terms of higher order than means and standard deviations are not needed to