

# A Bayesian Approach to Reliability-Based Optimization With Incomplete Information

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*In engineering design, information regarding the uncertain variables or parameters is usually in the form of finite samples. Existing methods in optimal design under uncertainty cannot handle this form of incomplete information; they have to either discard some valuable information or postulate existence of additional information. In this article, we present a reliability-based optimization method that is applicable when information of the uncertain variables or parameters is in the form of both finite samples and probability distributions. The method adopts a Bayesian binomial inference technique to estimate reliability, and uses this estimate to maximize the confidence that the design will meet or exceed a target reliability. The method produces a set of Pareto trade-off designs instead of a single design, reflecting the levels of confidence about a design's reliability given certain incomplete information. As a demonstration, we apply the method to design an optimal piston-ring/cylinder-liner assembly under surface roughness uncertainty. [DOI: 10.1115/1.2204969]*

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## 1 Introduction

Reliability-based design optimization (RBDO) is a loosely used term to refer to a class of methods in design under uncertainty that uses probabilistic quantification of the uncertain quantities (variables or parameters). Examples of these methods include: first/second order reliability method [1,2], first order second moment [3,4], adaptive importance sampling [5], advance mean value [6] and its hybrid variant [7], sequential optimization and reliability assessment [8], and single-loop method [9]. Broadly speaking, RBDO is just one class of methods within the umbrella of design under uncertainty. Another class is the interval-based class, which as its name suggests, quantifies the uncertain quantities as intervals. Examples of interval-based methods are: interval arithmetic [10], tolerance box [11], multi-point approximation [12], anti-optimization [13], and sensitivity region [14]. This classification is more or less a partition of the methods according to the level of information available in the design. RBDO is applicable when there is complete information of the uncertain quantities (infinite amount of data in terms of probability distributions). Interval-based methods are applicable to problems with incomplete information where behavior of the uncertain quantities within the bounds is not known.

While mathematically sound, these quantifications do not capture the reality of engineering design practice. In actual design, much of the information regarding the uncertain quantities is available in the form of a set of finite samples, either from historical data or from field measurements. These samples are usually not enough to infer probability distributions; yet to collapse them into intervals means discarding much of this valuable information. Also, collecting more samples is often not possible due to cost or time limitations. In terms of its information content, quantification of uncertainty as a set of finite samples can be considered a medium level of information (Fig. 1).

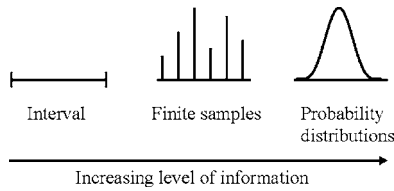
In this situation of incomplete information, neither RBDO nor interval analysis fully utilize the available information. There have

been efforts to develop a method that incorporates incomplete information in its uncertainty quantification, or even further, a method general enough as to cover a wide spectrum of the information content line. One approach focuses on reliability-based optimization under a mixture of both random and interval variables [15]. This method is quite successful in accounting for either end of the spectrum although it is not applicable to problems with finite samples. Another approach concentrates on inferring probability density functions (pdf's) of the uncertain quantities from a small set of samples so as to make the RBDO algorithms applicable [16,17]. The principle of maximum entropy [18] has also been used to derive a pdf based on the given data. Inferring a pdf, however, generates error in the data used and it is difficult to propagate this error to the error in the reliability prediction.

One research direction of increasing interest is the possibility-based method that uses fuzzy sets to quantify the uncertain quantities [19,20]. This approach is in line with the evidence theory approach that encompasses beliefs and plausibility [21]. It is argued that fuzzy sets can capture varying levels of information by controlling the  $\alpha$  cut of the membership functions. However, constructing a membership function is difficult since it involves "expert opinions" that may be different for each expert and may even be conflicting. There is an attempt too to merge the possibility quantification with a probability one so as to reap the benefits of both representations [22]. However, it is not yet clear what the analysis results signify since possibility and probability theories are based on entirely different axioms. An excellent comparison between possibility and probability-based approaches is given by Nikolaidis et al. [23].

Of all the methods discussed, none fully addresses the presence of incomplete information in practical design situations. We have to either discard some of the known information or hypothesize additional one. Our objective in this article is to develop a reliability-based optimization method for problems with incomplete information. More specifically, we utilize Bayesian statistics to solve problems for which a mixture of finite samples and pdf's of the uncertain quantities is known. The work presented in this article is an early stage of our research in this direction. The end aim is to include the notion of "value of information" into the framework of reliability-based optimization with incomplete in-

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**Fig. 1 Uncertainty quantification based on information content**

formation. The more information we have the better our predictive ability becomes, but it also costs more to collect additional information.

The detailed formulation and scope of the work is given in Sec. 2. Discussion of the method is divided into two parts: reliability analysis in Sec. 3, and its use in a reliability-based optimization setting in Sec. 4. As a demonstration, the method is applied to an engineering case study in Sec. 5. We conclude with a brief summary in Sec. 6.

## 2 Formulation and Scope

A general formulation of a deterministic optimization problem is shown in Eq. (1).

$$\underset{\mathbf{x}}{\text{minimize}} f(\mathbf{x}, \mathbf{p})$$

$$\text{subject to: } g_j(\mathbf{x}, \mathbf{p}) \leq 0, \quad j = 1, \dots, J \quad (1)$$

In this formulation,  $\mathbf{x}$  are the  $n$  design variables that are varied in the optimization while  $\mathbf{p}$  are the  $m$  design parameters that are kept fixed. Both  $\mathbf{x}$  and  $\mathbf{p}$  are assumed continuous. The objective is to minimize a function  $f$  subject to  $J$  inequality constraints  $\mathbf{g}$ . Equality constraints are omitted because reliability in this context is not defined for equality constraints.

In reliability-based optimization, uncertainties in the design are embodied as random design variables  $\mathbf{X}$  and random design parameters  $\mathbf{P}$ , Eq. (2).

$$\underset{\boldsymbol{\mu}_X}{\text{minimize}} f(\boldsymbol{\mu}_X, \boldsymbol{\mu}_P)$$

$$\text{subject to: } \Pr[g_j(\mathbf{X}, \mathbf{P}) \leq 0] \geq R_j, \quad j = 1, \dots, J$$

$$\text{where: } \mathbf{X} = [\mathbf{X}_t, \mathbf{X}_s] \text{ and } \mathbf{P} = [\mathbf{P}_t, \mathbf{P}_s] \quad (2)$$

The objective of the problem is to minimize  $f$  with respect to the means of the random variables  $\boldsymbol{\mu}_X$  given the means of the random parameters  $\boldsymbol{\mu}_P$ . The problem is subject to the constraints that the probability of design feasibility is greater than or equal to some value  $R_j$ , for all  $j = 1, \dots, J$ . The quantity  $R_j$  is the target reliability for the  $j$ th probabilistic constraint. Following conventional notations, the  $j$ th probabilistic constraint can also be written as  $F_{g_j}(0) \geq R_j$ , where  $0 \leq F_{g_j}(\cdot) \leq 1$  is the cumulative density function (cdf) of  $g_j$ . Solution to a reliability-based optimization problem is called the optimal-reliable design. In this article all random quantities are assumed independent.

The focus of this article is on the impact of uncertainty to the feasibility of a design; in other words, on what we call the "reliability" of a design. The impact of uncertainty to the performance (the objective function value) of a design falls into the subject of "robust design" and is beyond the scope of this article. For this reason, the objective function  $f$  in Eq. (2) is formulated simply as a function of the mean values of  $\mathbf{X}$  and  $\mathbf{P}$ .

To model the different levels of information available about the random variables and parameters, we partition them into two vectors:  $\mathbf{X} = [\mathbf{X}_t, \mathbf{X}_s]$  and  $\mathbf{P} = [\mathbf{P}_t, \mathbf{P}_s]$ . The vectors  $\mathbf{X}_t$  and  $\mathbf{P}_t$  are those random variables and parameters whose pdf's are known. This implies complete information about these random quantities. The vector  $\mathbf{X}_t = [X_{t,1}, \dots, X_{t,n_t}]$  consists of  $n_t$  random variables, and

each  $X_{t,i}$  is distributed according to a pdf  $f_{X_{t,i}}$ . Similarly, the vector  $\mathbf{P}_t = [P_{t,1}, \dots, P_{t,m_t}]$  has  $m_t$  components, and the pdf of each  $P_{t,i}$  is  $f_{P_{t,i}}$ . The vectors  $\mathbf{X}_s$  and  $\mathbf{P}_s$ , on the other hand, are those random variables and parameters whose pdf's are not known. Rather, only some samples are available, which implies incomplete information of these random quantities. In this article we assume that data are available collectively in that  $\mathbf{X}_s$  and  $\mathbf{P}_s$  are assumed to be available as  $N$  samples of  $(n_s + m_s)$  tuples. So one measurement of  $(X_{s,1}, \dots, X_{s,n_s}, P_{s,1}, \dots, P_{s,m_s})$  all together amounts to one sample. The vectors  $\mathbf{X}_s = [X_{s,1}, \dots, X_{s,n_s}]$  and  $\mathbf{P}_s = [P_{s,1}, \dots, P_{s,m_s}]$  have  $n_s$  and  $m_s$  components, respectively. Accordingly,  $n = n_t + n_s$  and  $m = m_t + m_s$ . A reliability-based optimization is said to have incomplete information if  $[\mathbf{X}_s, \mathbf{P}_s] \neq \emptyset$ .

The presence of both pdf's and finite samples in the formulation prohibits the use of conventional methods to calculate the probabilistic constraints in Eq. (2), and hence to obtain the optimal-reliable solution. The next section presents a method to estimate these constraints using Bayesian inference.

## 3 Reliability Estimation

Let  $\Pr[g_j(\mathbf{X}, \mathbf{P}) \leq 0] = F_{g_j}(0)$  be the probabilistic constraint of interest. If the pdf's of all  $\mathbf{X}$  and  $\mathbf{P}$  are known (complete information), then they can be propagated through  $g_j$  to obtain the one value of  $F_{g_j}(0)$ . If some of these pdf's are not known, however, then it would not be possible to calculate  $F_{g_j}(0)$  precisely. Under this incomplete information condition, at best we can only expect to obtain a range of estimates for  $F_{g_j}(0)$ , where possibly some values are more likely than others; in other words, a distribution estimate of  $F_{g_j}(0)$  instead of a point estimate. The following sections show the use of a Bayesian inference technique to calculate this distribution estimate and to update the estimate given more information.

**3.1 Bayesian Binomial Inference.** Consider a Bernoulli process, such as a coin toss, whose probabilities of "success" and "failure" are  $p$  and  $(1-p)$ , respectively. Given  $N$  independent trials, the probability of having  $r$  successes out of these trials follows a Binomial distribution:  $r \sim \text{Bin}(N, p)$ .

In a Binomial distribution, the probability of success  $p$  of each trial is known, and we want to predict the outcome of a trial from it. An inference of this process seeks to calculate  $p$  based on the outcomes of the trials. Given  $r$  successes out of  $N$  trials, the probability distribution of  $p$  can be calculated using Bayes' theorem shown in Eq. (3)

$$f(p|r) = \frac{f(p) \times f(r|p)}{\int_0^1 f(p) \times f(r|p) dp} \quad (3)$$

In this equation  $f(p)$  is the prior distribution of  $p$ ,  $f(p|r)$  is the posterior distribution of  $p$ , and  $f(r|p)$  is the likelihood of  $r$  given  $p$ . The integral in the denominator is a normalizing factor to make the probability distribution proper.

The posterior distribution is the distribution of interest. It is the estimate of  $p$  based on the outcome of the trials. The prior distribution is our knowledge about  $p$  before the information from the trials. Many types of prior distribution have been proposed, but it has been shown that the analysis is insensitive to the choice of prior if the number of trials is large enough. In this article we assume a uniform prior for  $p$ . This assumption implies that any values of  $p$  between 0 and 1 are equally probable. It is a conservative choice based on the maximum ignorance principle. The only information we have a priori is that  $p$  must be within  $[0, 1]$ , and a uniform distribution can be fully defined based solely on this information. The likelihood function is the probability that there will be  $r$  successes out of  $N$  trials if the probability of suc-

cess is  $p$ , which is the binomial distribution of  $r$ .

Using a uniform prior and a Binomial likelihood function in Eq. (3) results in a Beta posterior distribution, Eq. (4), where  $\alpha=r+1$  and  $\beta=(N-r)+1$ .

$$f(p|r) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \quad (4)$$

In other words,  $p$  is distributed according to a Beta distribution whose two parameters depend on the outcome of the trials:  $p \sim \text{Beta}(r+1, (N-r)+1)$ .

One very important feature of Bayes' theorem is that it facilitates an updating scheme to account for additional information. Suppose that after the  $N_1$  initial trials, we conduct  $N_2$  additional trials and observe  $r_2$  more successes. In Bayes' theorem, the posterior distribution from the  $N_1$  trials can be used as the prior distribution for the  $N_2$  trials, thus creating a chain of analysis based on additional information. Using  $\text{Beta}(\alpha_1, \beta_1)$  for the prior distribution in Eq. (3), the posterior distribution of  $p$  after the additional  $N_2$  trials is as shown in Eq. (5). Here  $\alpha_2 = \alpha_1 + r_2$  and  $\beta_2 = \beta_1 + (N_2 - r_2)$ .

$$f(p|r_2) = \frac{\Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} p^{\alpha_2-1}(1-p)^{\beta_2-1} \quad (5)$$

It can be immediately seen that the new posterior is also a Beta distribution with the two parameters equal to the prior parameters updated by the new information. This is a conjugacy feature of a Beta distribution when used in Bayes' theorem. This conjugacy is not limited to just two successive trials. If there is a sequence of  $(N_1, N_2, \dots, N_k)$  trials, then the posterior of the  $(k-1)$ th trial becomes the prior of the  $k$ th trial, except for  $k=1$  whose prior is  $\text{Beta}(1,1)$  (mathematically, a uniform distribution is equivalent to a  $\text{Beta}(1,1)$  distribution). In general, the posterior distribution of  $p$  at the  $k$ th trial is  $\text{Beta}(\alpha_k, \beta_k)$  where  $\alpha_k = \alpha_{k-1} + r_k$  and  $\beta_k = \beta_{k-1} + (N_k - r_k)$ ,  $k \geq 1$ .

The discussion in this section is only a very brief summary of Bayesian inference. For more details, readers should consult references on Bayesian statistics, for example, [24–26]. In the next section, this inference technique is used to obtain the  $F_{g_j}(0)$  distribution estimate.

**3.2 Bayesian Reliability Estimate.** In an optimization problem, an inequality constraint  $g_j$  divides the design space into two domains: feasible and infeasible. Suppose for now that the pdf's of all  $\mathbf{X}$  and  $\mathbf{P}$  are unknown:  $[\mathbf{X}_r, \mathbf{P}_r] = \emptyset$ , and that there are only  $N$  samples available. Each of these samples is a realization of  $(\mathbf{X}, \mathbf{P})$  in the design space, so each sample corresponds to either a feasible or an infeasible realization. Let  $r$  and  $(N-r)$  be the number of feasible and infeasible realizations, respectively. The goal is to estimate  $F_{g_j}(0)$  given these  $r$  and  $(N-r)$  realizations.

This  $F_{g_j}(0)$  estimation problem is analogous to the Bayesian inference problem discussed previously. If we use a coin toss as an analogy, then  $g_j$  is the "coin" with only two mutually exclusive outcomes, the  $N$  samples are the "coin tosses" or trials, and  $F_{g_j}(0)$  is the "probability of head," where we have designated a feasible realization as a "head." The problem of inferring  $F_{g_j}(0)$  given  $r$  and  $(N-r)$  realizations is then equivalent to the problem of inferring  $p$ , the probability of heads, given that there are  $r$  heads out of  $N$  coin tosses. From the above discussion we obtain  $F_{g_j}(0) \sim \text{Beta}(\alpha, \beta)$ , where  $\alpha=r+1$  and  $\beta=(N-r)+1$ .

A Beta distribution is strictly bounded within  $[0,1]$ . This makes the estimate proper because  $F_{g_j}(0)$  is a cdf. The shape of a Beta distribution depends on the number of feasible-infeasible realizations. If there are many feasible realizations, it will be skewed to the right towards one; if there are many infeasible realizations, it

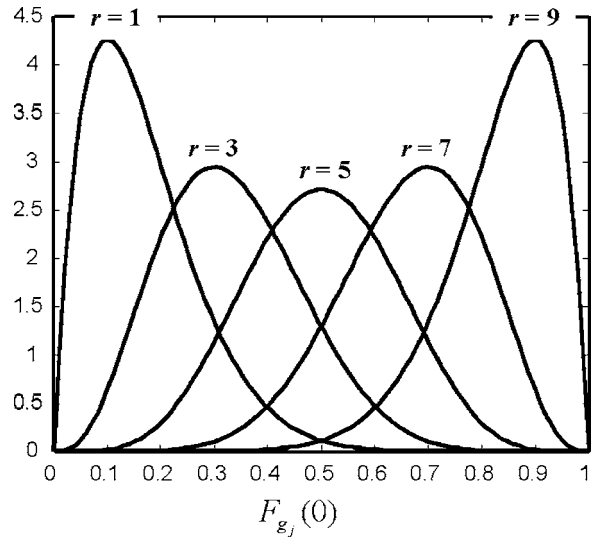


Fig. 2 Beta-distributed  $F_{g_j}(0)$  estimate for different  $r$  ( $N=10$ )

will be skewed to the left towards zero (Fig. 2). This is also a proper behavior for a  $F_{g_j}(0)$  estimate. If a design has many feasible realizations, we expect its reliability to be high, and vice versa.

This estimate of  $F_{g_j}(0)$  assumes that the pdf's of all  $\mathbf{X}$  and  $\mathbf{P}$  are unknown. When some of the pdf's are known ( $[\mathbf{X}_r, \mathbf{P}_r] \neq \emptyset$ ), the coin toss analogy still holds provided that this pdf information is accounted for. The difference between the two cases lies in the calculation of the feasible-infeasible realization of a design. In the case where all pdf's are unknown, one  $(\mathbf{X}_s, \mathbf{P}_s)$  sample gives a distinct feasible or infeasible realization, either to the left or right of  $g_j=0$  in the  $g_j$  axis. In contrast, when some of the pdf's are known, one sample results in a distribution of  $g_j$  values (Fig. 3).

Since each  $(\mathbf{X}_s, \mathbf{P}_s)$  sample now has a distribution of  $g_j$  values, it is not possible to simply count the number of feasible or infeasible realizations. What we can calculate instead is the probability that  $g_j$  is feasible given the  $k$ th sample of  $(\mathbf{X}_s, \mathbf{P}_s)$ :  $\Pr[g_j(\mathbf{X}_s, \mathbf{P}_s) \leq 0 | (\mathbf{X}_s, \mathbf{P}_s)_k]$ . This probability is the expected feasible realization of one sample. The sum of the probabilities of all samples is then the expected total number of feasible realizations of the design

$$E(r) = \sum_{k=1}^N \Pr[g_j(\mathbf{X}_s, \mathbf{P}_s) \leq 0 | (\mathbf{X}_s, \mathbf{P}_s)_k] \quad (6)$$

Using  $E(r)$  as the expected number of heads, we can use the coin toss analogy again to estimate  $F_{g_j}(0)$ . This new estimate is given by a Beta distribution with  $\alpha=E(r)+1$  and  $\beta=(N-E(r))+1$ . Notice that Eq. (6) is valid for both when only  $(\mathbf{X}_s, \mathbf{P}_s)$  samples are available and when there is a mix of  $(\mathbf{X}_s, \mathbf{P}_s)$  samples

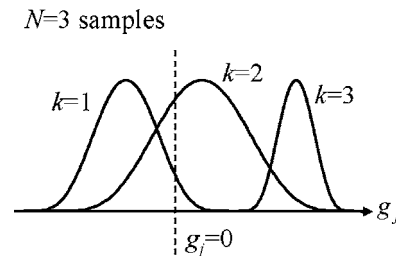


Fig. 3 Feasible-infeasible realization of a  $(\mathbf{X}_s, \mathbf{P}_s)$  sample given  $(\mathbf{X}_r, \mathbf{P}_r)$  pdf's

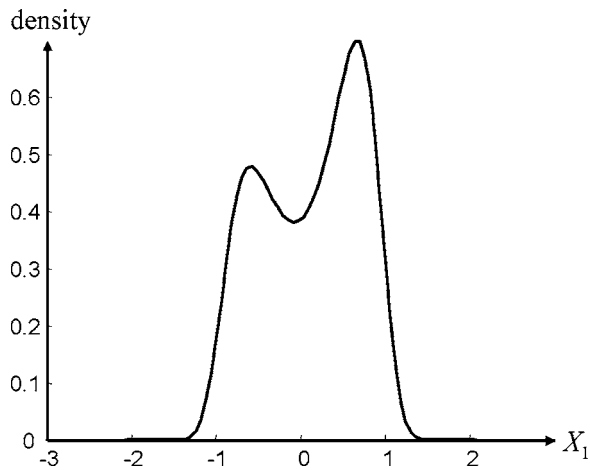


Fig. 4 Probability distribution of  $X_1$

and  $(\mathbf{X}_r, \mathbf{P}_r)$  pdf's. In the case where there is no known pdf, each probability in Eq. (6) becomes an indicator function where  $I_k=1$  if  $g_j(\mathbf{X}_s, \mathbf{P}_s)_k \leq 0$ , and  $I_k=0$  otherwise.

Under incomplete information,  $F_{g_j}(0)$  can be estimated with a Beta distribution. The precision of this estimate (graphically depicted as the spread of the distribution) depends on the number of samples. As  $N$  increases, the estimate becomes more precise. In the extreme that  $N \rightarrow \infty$ , the distribution converges to a Dirac delta function meaning that  $F_{g_j}(0)$  is known exactly, which is the same as knowing the pdf's of all  $\mathbf{X}$  and  $\mathbf{P}$ . This link between  $N$  and precision of the estimate is captured by the Bayesian update. Following previous discussion, adding  $N_2$  samples to an existing  $N_1$  samples revises the  $F_{g_j}(0)$  estimate from a  $\text{Beta}(\alpha_1, \beta_1)$  to a  $\text{Beta}(\alpha_2, \beta_2)$ , where  $\alpha_2 = \alpha_1 + E(r_2)$  and  $\beta_2 = \beta_1 + (N_2 - E(r_2))$ . This updating scheme will be demonstrated next with an example.

**3.3 Reliability Estimation Example.** Let  $g(X_1, X_2) \equiv 0.3X_1^2X_2 - X_2 + 0.8X_1 + 1 \leq 0$  be an inequality constraint with two random variables. The random variable  $X_2$  is normally distributed around its mean with a given standard deviation:  $X_2 \sim N(\mu_{X_2}, 0.5)$ . The random variable  $X_1$  is distributed according to a pdf  $f_{X_1} = \exp(-2.5X_1^4 + 2X_1^2 + 0.3X_1)/2.586$  as shown in Fig. 4. The pdf shown, for which  $\mu_{X_1} = 0.111$ , is considered to be the "standard" pdf of  $X_1$ . The distribution of  $X_1$  at other mean values is obtained by shifting this pdf accordingly.

For demonstration purposes, let us assume that the multi-modal pdf of  $X_1$  is not actually known. Rather, we are provided only with  $N=10$  values sampled randomly from this pdf. The normal distribution of  $X_2$  is assumed known. So in this problem  $\mathbf{X}_r = [X_2]$  and  $\mathbf{X}_s = [X_1]$ , and  $\mathbf{P}_r = \mathbf{P}_s = \emptyset$ . Our goal is to estimate the reliability of a design  $(\mu_{X_1}, \mu_{X_2}) = (-2, 0.5)$  with respect to  $g$  given the incomplete information of  $N$  samples of  $X_1$  and a pdf of  $X_2$ .

This problem involves a mix of pdf and samples. So to obtain the Beta distribution of  $F_g(0)$ , we first need to calculate the probability  $\text{Pr}[g(X_2) \leq 0 | X_{1,k}]$  for all  $k=1, \dots, 10$ . For instance, for the

Table 1  $X_1$  samples and their probabilities

$X_1$	Pr	$X_1$	Pr
-1.825	1.0	-2.421	0.929
-1.624	0.999	-1.594	0.999
-2.385	0.941	-1.176	0.7878
-2.575	0.873	-2.311	0.965
-2.122	0.998	-1.783	1.0

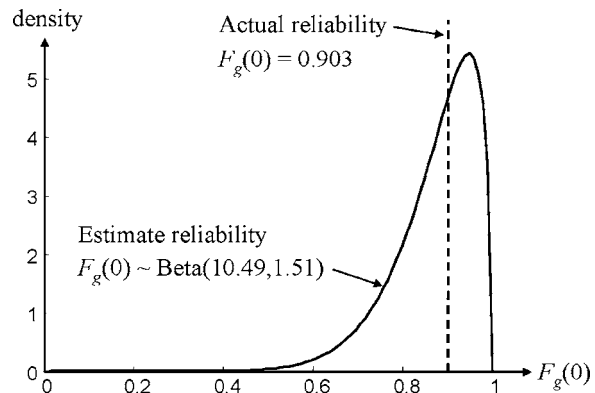


Fig. 5 Actual and estimate reliability of  $(\mu_{X_1}, \mu_{X_2}) = (-2, 0.5)$

sample  $X_1 = -2.575$ ,  $\text{Pr}[g(X_2) \leq 0 | X_1 = -2.575] = 0.873$ . This probability can be calculated with direct integration since it is only one dimensional. Table 1 shows the ten  $X_1$  samples and their associated probability values.

The sum of all the probabilities in Table 1 is  $E(r) = 9.49$ . This is the expected total number of feasible realizations of the design point. Using  $E(r) = 9.49$  and  $N - E(r) = 0.51$ , the estimate of  $F_g(0)$  is then  $\text{Beta}(E(r) + 1, N - E(r) + 1) = \text{Beta}(10.49, 1.51)$ , as shown in Fig. 5. The actual reliability of the design point calculated using the multi-modal pdf of  $X_1$  and the normal pdf of  $X_2$  is  $F_g(0) = 0.903$ . As seen in Fig. 5, this is not a very precise estimate because there are relatively few samples to work with. The imprecision reflects the low level of information available.

Let us now look at the effect of adding more  $X_1$  samples to the problem. Suppose after the first ten samples, we add 90 more samples so that in total  $N = 100$ . The expected total number of feasible realizations of the 90 new samples is  $E(r_{90}) = 80.04$ . Following the Bayesian update, the parameters of the new Beta distribution are  $\alpha_{100} = \alpha_{10} + E(r_{90}) = 90.53$  and  $\beta_{100} = \beta_{10} + (90 - E(r_{90})) = 11.47$ . This updating scheme can be repeated as more samples are added. Figure 6 shows the Beta distributions of  $F_g(0)$  for  $N = 10, 100$ , and 500.

We observe in Fig. 6 that additional samples increase the precision of the estimate. In the limit that  $N \rightarrow \infty$ , the distribution estimate converges to the actual  $F_g(0)$  value, which is the same as knowing the multi-modal pdf of  $X_1$ . Next we use this Bayesian reliability estimation method to solve a reliability-based optimization problem.

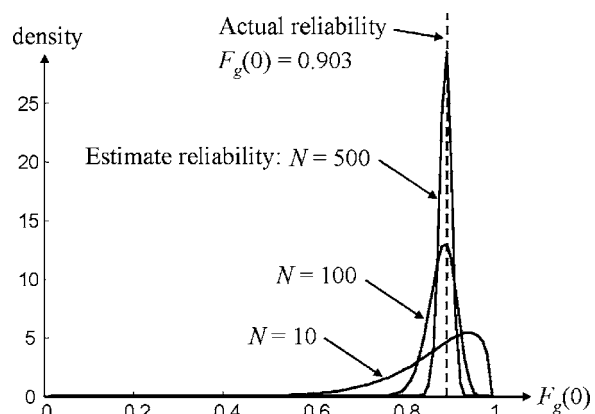


Fig. 6 Beta distribution of  $F_g(0)$  for increasing  $N$

## 4 Reliability-Based Optimization

Because of the incomplete information about the random variables/parameters, we can only obtain a distribution estimate of a design's reliability instead of a single value. Consequently, in the design optimization we cannot state with certainty that a design's reliability is smaller or larger than the target value. There is not enough information to make such a precise statement. The optimization then requires an additional measure to decide if a design can be deemed to satisfy the reliability requirement. We define a quantity we call confidence to be such as a measure. For a design, the confidence of that design with respect to the  $j$ th inequality constraint is defined to be the probability that it will meet or exceed the reliability target.

$$\zeta_j(\boldsymbol{\mu}_X) = \Pr[F_{g_j}(0) | \boldsymbol{\mu}_X \geq R_j]; \quad j = 1, \dots, J \quad (7)$$

A  $\zeta_j=0$  means that the design is certainly not reliable, while a  $\zeta_j=1$  means that the design certainly meets or exceeds the target. Since the  $F_{g_j}(0)$  estimate is a Beta distribution, the confidence can also be written as  $\zeta_j(\boldsymbol{\mu}_X) = 1 - \Phi_{B_j}(R_j)$  where  $\Phi_{B_j}(\cdot)$  is the cdf of the  $j$ th Beta distribution, for all  $j=1, \dots, J$ .

When we have complete information regarding the random variables/parameters, optimization seeks a design with the best objective value and with reliability greater than or equal to  $R_j$ . In the incomplete information case, this condition no longer applies since the constraint satisfaction  $F_{g_j}(0) \geq R_j$  is only known as a probability (i.e., only  $\zeta_j$  is known) instead of a crisp yes/no decision. The optimization problem then becomes ill defined in that there is no one solution that is "best" with respect to the objective value only. A design can have a very good value of  $f$  but poor probability of meeting the reliability target, and vice versa. In other words, there is a trade-off between minimizing  $f$  and satisfying the reliability constraints.

To account for this trade-off, the problem must be reformulated into a multi-objective optimization problem where the objectives are to minimize  $f$  and to maximize the  $\zeta_j$ 's. There is a  $J$  number of  $\zeta_j$ ; however, in reliability-based optimization a design is considered feasible if all probabilistic constraints are satisfied. Following this convention, the  $\zeta_j$ 's can be lumped into:  $\zeta_s(\boldsymbol{\mu}_X) = \min_{j=1, \dots, J} \zeta_j(\boldsymbol{\mu}_X)$ , a quantity called the *overall confidence* of a design. Using this measure the multi-objective problem then becomes as shown in Eq. (8).

$$\begin{aligned} & \underset{\boldsymbol{\mu}_X}{\text{minimize}} && f(\boldsymbol{\mu}_X, \boldsymbol{\mu}_P) \\ & \underset{\boldsymbol{\mu}_X}{\text{maximize}} && \zeta_s(\boldsymbol{\mu}_X) \\ & \text{subject to:} && 0 \leq \zeta_s(\boldsymbol{\mu}_X) \leq 1 \end{aligned} \quad (8)$$

Solving Eq. (8) will in general result in a set of Pareto optima instead of a single optimal design [27]. This is a consequence of having incomplete information about  $\mathbf{X}$  and  $\mathbf{P}$ : it is not possible to pinpoint exactly which design is both optimal and reliable. Rather, we can only obtain a set of designs, each with a certain degree of probability of being the true optimal-reliable design (one that corresponds to complete information).

The multi-objective formulation in Eq. (8) makes it possible for the Pareto solutions to undershoot or overshoot the true optimal-reliable design in terms of the objective  $f$ . This is because a design that is previously infeasible with respect to the probabilistic constraints can have a non-zero  $\zeta_s$  if it is not too far from the probabilistic boundaries. If the problem is properly bounded and is locally monotonic [28] in that relaxing the active probabilistic constraints improves the optimal  $f$  value, then the farther a design is from the active boundaries towards the infeasible domain, the better it becomes in terms of  $f$  but worse in terms of  $\zeta_s$ . Designs with this trade-off form part of the Pareto frontier and their  $f$  values overshoot that of the true optimal-reliable design. The re-

verse is also possible. The farther a design is from the active boundaries towards the feasible domain, the better it becomes in terms of  $\zeta_s$  but worse in terms of  $f$ . A portion of the Pareto frontier formed by these designs undershoots the  $f$  value of the true optimal-reliable design.

If the information of the random quantities is only available as samples, i.e.,  $[\mathbf{X}_r, \mathbf{P}_r] = \emptyset$ , the Pareto frontier of Eq. (8) will be necessarily discrete. This property can be explained as follows. The Beta distribution of the  $F_{g_j}(0)$  estimate is created from updating a uniform prior distribution with the number of feasible-infeasible realizations. For  $N$  samples, the possible  $r$  values are discrete from a minimum of 0 to a maximum of  $N$ . So there are only a total of  $(N+1)$  different Beta distributions possible (recall Fig. 2), and this translates to only  $(N+1)$  discrete possible values of  $\zeta_s$ . Since  $\zeta_s$  is one of the two objectives in Eq. (8), the Pareto frontier can only contain these particular values, and hence is discrete. If the pdf's of some of  $\mathbf{X}$  and  $\mathbf{P}$  are known, the discreteness of  $\zeta_s$  disappears. However, the Pareto frontier may be discrete or continuous depending on the continuity of  $f$ .

**4.1 The  $N$ - $R$ - $\zeta_s^{\max}$  Relationship.** Regardless of whether or not the Pareto frontier is discrete, it is clear that there is an upper bound to the  $\zeta_s$  value of the frontier. This maximum attainable confidence,  $\zeta_s^{\max}$ , of the Pareto solutions depends on the number of samples and the reliability target. Intuitively, higher  $N$  implies higher  $\zeta_s^{\max}$  and higher  $R$  implies lower  $\zeta_s^{\max}$ . This relationship between  $N$ ,  $R$  and  $\zeta_s^{\max}$  is a valuable information for a decision maker as it provides a trade-off between how much confidence can be achieved by adding more information, which costs more to collect, and by relaxing the reliability target, which degrades quality.

Let  $R_j=R$  be the same for all  $j=1, \dots, J$ . If  $R_j$  is different for each  $j$ , the  $\zeta_s^{\max}$  corresponds to  $R^{\min} = \min_{j=1, \dots, J} R_j$ . Also, for generality let us consider a problem with a mix of samples and pdf's. As before, the Beta distribution of the  $F_{g_j}(0)$  estimate is created from updating a uniform prior distribution with  $E(r)$  and  $(N-E(r))$ . From the infinite number of possible Beta distributions, the right-most distribution is one whose  $E(r)=N$  or, in other words, it is a Beta distribution whose  $\alpha=N+1$  and  $\beta=1$ . This right-most Beta distribution is the highest reliability estimate achievable, and this distribution corresponds to  $\zeta_s^{\max}$ . Following the definition of confidence, we obtain  $\zeta_s^{\max} = 1 - \Phi_B(R | \alpha=N+1, \beta=1)$ .

This equation can be simplified by substituting the  $\alpha=N+1$  and  $\beta=1$  values into the equation of a Beta pdf and integrating it from 0 to  $R$ .

$$\zeta_s^{\max} = 1 - \int_{x=0}^R \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx \quad (9)$$

The constant  $\Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta)) = \Gamma(\alpha+1)/(\Gamma(\alpha)\Gamma(1))$  reduces to just  $\alpha=N+1$  because  $\Gamma(1)=1$  and  $\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha$ . The second term in the integration becomes  $(1-x)^{\beta-1} = 1$  for  $\beta=1$ . Equation (9) then becomes

$$\zeta_s^{\max} = 1 - (N+1) \int_{x=0}^R x^N dx = 1 - R^{N+1} \quad (10)$$

This simple equation links the three quantities of interest  $N$ ,  $R$ , and  $\zeta_s^{\max}$  together. Figure 7(a) shows the  $N$ - $R$ - $\zeta_s^{\max}$  diagram with  $\zeta_s^{\max}$  as the ordinate for several values of  $N$ . In practice, it may be more useful to determine  $N$  from  $R$  and  $\zeta_s^{\max}$  as shown in Fig. 7(b) where  $N = \log_R(1 - \zeta_s^{\max}) - 1$ .

We observe in these diagrams that overall, increasing  $N$  or decreasing  $R$  will increase  $\zeta_s^{\max}$ . At the one extreme where  $R=0$ , we have  $\zeta_s^{\max}=1$  and  $N=0$ , implying that no sample is needed. Any Beta distribution will always be to the right of  $R=0$  so clearly

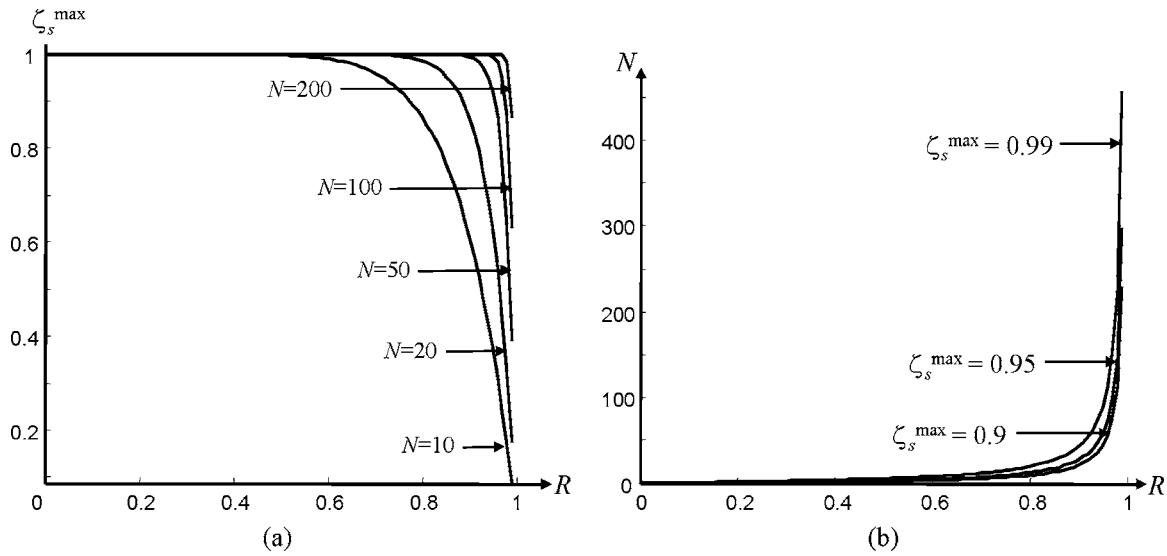


Fig. 7 The  $N$ - $R$ - $\zeta_s^{\max}$  diagram; (a)  $\zeta_s^{\max}$  as ordinate, (b)  $N$  as ordinate

$\zeta_s^{\max}=1$ . At the other extreme where  $R=1$ ,  $\zeta_s^{\max}$  is either zero or 1 with no value in between. For this case  $\zeta_s^{\max}=1$  only if  $N \rightarrow \infty$ , meaning that we need to know the actual pdf's of all  $\mathbf{X}$  and  $\mathbf{P}$ . This is reasonable because for  $R=1$  any Beta distribution will be to the left of it and hence  $\zeta_s^{\max}=0$ , unless the distribution converges to the Dirac delta function, which implies knowing the pdf's (complete information).

**4.2 Behavior of the Pareto Frontier.** The existence of the upper bound  $\zeta_s^{\max}$  and its relationship with  $N$  and  $R$  govern the behavior of the Pareto frontier when more samples are added or when the reliability target is relaxed. If  $N$  and  $R$  are such that  $\zeta_s^{\max} < 1$ , the addition of  $\Delta N$  samples increases the  $\zeta_s^{\max}$  of the Pareto frontier by:

$$\Delta \zeta_s^{\max} |_{\Delta N} = \zeta_{s, N+\Delta N}^{\max} - \zeta_{s, N}^{\max} \quad (11)$$

$$\Delta \zeta_s^{\max} |_{\Delta N} = (1 - R^{(N+\Delta N)+1}) - (1 - R^{N+1}) \quad (12)$$

or more compactly  $\Delta \zeta_s^{\max} |_{\Delta N} = (1 - R^{\Delta N})R^{N+1}$ . Relaxing the reliability target to  $R_2 = \theta_R R$  where  $0 \leq \theta_R < 1$  given a fixed  $N$  also increases the  $\zeta_s^{\max}$  of the Pareto frontier by  $\Delta \zeta_s^{\max} |_{\theta_R} = (1 - \theta_R^{N+1})R^{N+1}$ , which is positive since  $0 \leq \theta_R < 1$ . If we set  $1 < \theta_R \leq 1/R$ ,  $\Delta \zeta_s^{\max} |_{\theta_R}$  will be negative; this quantifies the decrease in  $\zeta_s^{\max}$  due to the increase in the reliability target requirement.

The decrease in  $\zeta_s^{\max}$  from increasing  $R$  can be compensated by increasing  $N$  at the same time. Thus, there is an equilibrium value for  $\Delta N$  and  $\theta_R$  such that in total  $\Delta \zeta_s^{\max} = 0$ . The condition for this equilibrium is as follows. The change in  $\zeta_s^{\max}$  due to  $\Delta N$  and  $R_2 = \theta_R R$  is  $\Delta \zeta_s^{\max} = R^{N+1} - (\theta_R R)^{N+\Delta N+1} = R^{N+1} [1 - (\theta_R^{N+1})(\theta_R R)^{\Delta N}]$ . For  $\Delta \zeta_s^{\max} = 0$ , either of the two terms has to be zero.  $R^{N+1} = 0$  only if  $R = 0$ . This implies a deterministic optimization, which is not of interest. Therefore, the second term must be zero. This is the equilibrium condition we seek

$$(\theta_R^{N+1})(\theta_R R)^{\Delta N} = 1 \quad (13)$$

After rearrangement

$$\theta_R^{N+\Delta N+1} = R^{-\Delta N} \quad (14)$$

Taking the logarithm of base  $\theta_R$  of both sides and converting the base  $\theta_R$  logarithm to a base  $R$  logarithm, the equation becomes

$$N + \Delta N + 1 = \frac{\log_R(R^{-\Delta N})}{\log_R(\theta_R)} \quad (15)$$

Rearranging the terms and taking the exponential of both sides with base  $R$ , we arrive at the final form of the equilibrium condition

$$\theta_R = R^{(-\Delta N/N+\Delta N+1)} \quad (16)$$

The change in  $N$  and  $R$  increases the  $\zeta_s^{\max}$  of the Pareto frontier only until  $\zeta_s^{\max} = 1$ . Recall Fig. 7 for necessary  $N$  and  $R$  values to reach this saturation. Preliminary results indicate that after  $\zeta_s^{\max} = 1$  is reached, additional samples beyond this threshold decrease the spread of the  $f$  values of the Pareto frontier in the objective space. Similar behavior is observed in the design variable space where the set of Pareto points forms a tighter cluster as more samples are added. However, further investigations are needed before any conclusions can be drawn. This is one issue under consideration in our ongoing research in this direction. We will demonstrate the application of this reliability-based optimization method with an example in the next section.

**4.3 Reliability-Based Optimization Example.** This example is a well-known test problem for reliability-based optimization [7–9] with two variables, a linear objective, and three nonlinear probabilistic constraints, shown in Eq. (17)

$$\text{minimize } f(\boldsymbol{\mu}_X) = \mu_{X_1} + \mu_{X_2}$$

$$\text{subject to: } \Pr[g_1: 1 - X_1^2 X_2 / 20 \leq 0] \geq R_1$$

$$\Pr \left[ \begin{array}{l} 1 - (X_1 + X_2 - 5)^2 / 30 \\ g_2: - (X_1 - X_2 - 12)^2 / 120 \leq 0 \end{array} \right] \geq R_2$$

$$\Pr[g_3: 1 - 80 / (X_1^2 + 8X_2 - 5) \leq 0] \geq R_3 \quad (17)$$

The variables are random and we assume that  $X_1$  follows the distribution  $f_{X_1} = \exp(-5X_1^4 + 1.5X_1^2 + 0.5X_1) / 1.614$  as shown in Fig. 8, while  $X_2$  follows a Beta( $\alpha=1.5, \beta=5$ ) distribution. As in the previous example, these pdf's are taken as the "standard" distributions. The pdf's of  $X_1$  and  $X_2$  at other mean values are obtained by shifting these distributions accordingly.

For demonstration purposes let us assume that the actual pdf of  $X_1$  is not known; rather, only  $N$  number of  $X_1$  samples taken from the actual pdf are known. The pdf of  $X_2$  is assumed known. So we

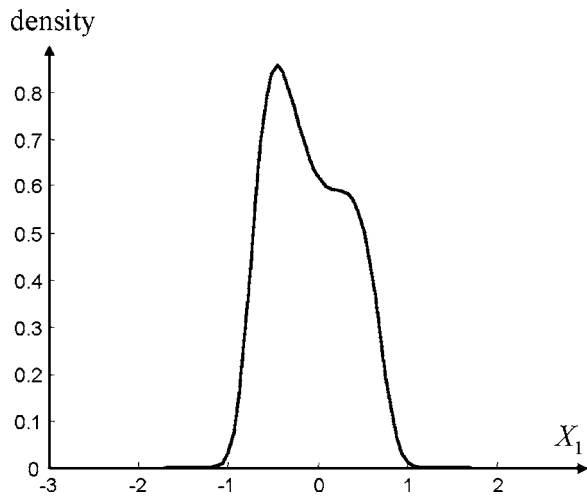


Fig. 8 Probability distributions of  $X_1$

have a reliability-based optimization problem with incomplete information of the random quantities:  $\mathbf{X}_T=[X_2]$  and  $\mathbf{X}_S=[X_1]$ , and  $\mathbf{P}_T=\mathbf{P}_S=\emptyset$ . We convert Eq. (17) into a multi-objective formulation shown in Eq. (8) and solve the resulting bi-objective problem. The reliability targets are set to be all equal:  $R_1=R_2=R_3=0.95$ . Figure 9 shows in the objective space the Pareto optima obtained for  $N=10, 25$ , and 135 samples. In this figure, the horizontal dash line is the  $\zeta_s^{\max}$  of the Pareto frontier while the vertical dash line is the  $f$  value of the true optimal reliable design calculated using the actual pdf of  $X_1$ :  $f^*=6.07$ .

We observe in Fig. 9 that there is a trade-off between the objective value of the optimal design and our confidence in its reliability. This trade-off is because there is incomplete information

about the random variables. The task of selecting which design to use is left to the designer and depends on his/her risk-taking behavior. Notice also in the figures how the Pareto optima are evolving towards  $f^*=6.07$  as  $N$  increases. The  $f$  values for  $N=10$  is spread out over the range  $[5.1, 6.05]$ , but this range shifts to  $[5.6, 6.2]$  and  $[5.9, 6.4]$  for  $N=25$  and 135, respectively. We also see in Fig. 9 that the  $\zeta_s^{\max}$  of the Pareto frontier increases as  $N$  increases; for  $N=10, 25$ , and 135, the  $\zeta_s^{\max}$  are 0.43, 0.74, and 1, respectively. Notice how  $\zeta_s^{\max}=1$  for  $N=135$ , which is the reason behind this unorthodox choice of  $N$  in our demonstration. These  $\zeta_s^{\max}$  values match very well with those predicted from Eq. (10) and the  $N$ - $R$ - $\zeta_s^{\max}$  diagram in Fig. 7.

Figure 10 shows the Pareto solutions obtained in the design variable space for  $N=10, 25$ , and 135. In this figure the two curves are the deterministic constraint boundaries with the arrows indicating feasible direction. For this problem only  $g_1$  and  $g_2$  are active while  $g_3$  is inactive. The direction of decreasing  $f$  is also shown. The square in this figure marks the true optimal-reliable design:  $(\mu_{X_1}, \mu_{X_2})^*=(3.4, 2.7)$ , and the cross marks are the Pareto solutions of the incomplete problem.

As can be seen in these figures, the Pareto solutions of the incomplete problem are evolving towards the true optimal-reliable solution. For low number of samples  $N=10$  and  $N=25$ , none of the Pareto solution is the true solution. This is a reflection of the non-unity confidence of these solutions,  $\zeta_s < 1$ . Only when  $N=135$  does the Pareto set contain the true solution. The  $\zeta_s=1$  of some of the points in the Pareto set imply that we are certain they will meet the reliability target based on the data provided, and this is reflected in the figure.

With regards to the given data, when  $N=135$  it may be possible to fit a probability distribution to these samples. However, one difficulty in this exercise is that we need to know the function for which to fit the data. The actual pdf of  $X_1$  in this particular example does not follow any of the conventional forms of probabil-

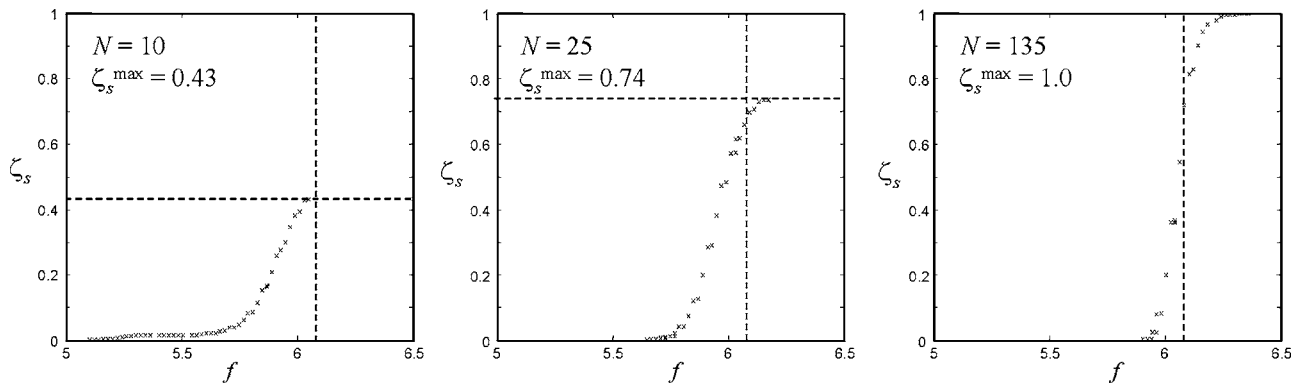


Fig. 9 Pareto optima for different sample numbers  $N$

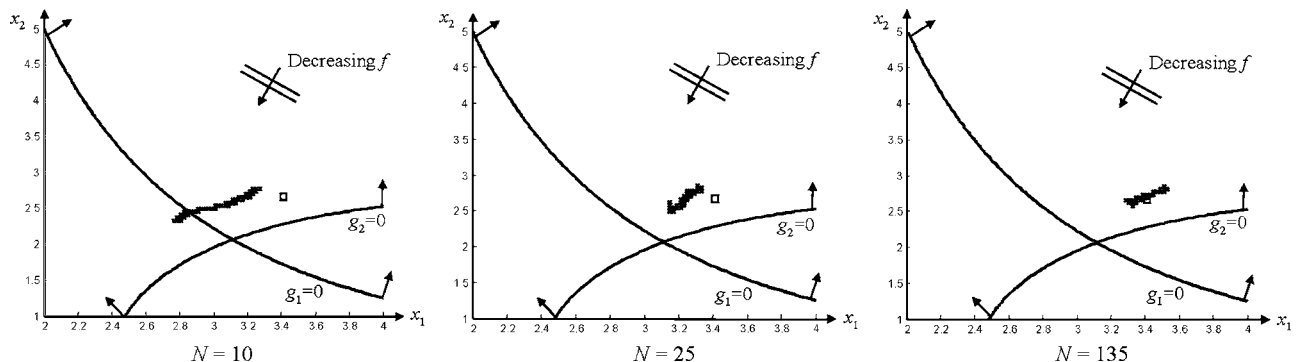


Fig. 10 Pareto optima in the design variable space

**Table 2 Maximum likelihood estimate of  $X_1$  samples**

$N$	$\hat{\mu}$	95% CI	$\hat{\sigma}$	95% CI
10	0.245	$\pm 0.328$	0.434	$\pm 0.401$
25	0.067	$\pm 0.185$	0.439	$\pm 0.184$
135	-0.074	$\pm 0.074$	0.436	$\pm 0.061$

ity distribution, so assuming we do not know the equation used in this example beforehand, deciding on which function to use is not a trivial task. Another difficulty is that even if the fit is reasonably acceptable, propagating the error incurred from this fitting to the error in the optimal-reliable design obtained is also not trivial.

To compare the solutions obtained from the Bayesian method with that obtained from the distribution fitting approach, let us fit a Gaussian distribution to the  $X_1$  samples (these are samples from the standard pdf of  $X_1$ ). A Gaussian distribution is chosen because it is commonly used in practice. It is used simply to make a comparison with the solutions obtained previously, and does not imply that it is the best fit to use. Table 2 shows the fitted parameters and their 95% confidence intervals obtained using the most likelihood estimate (MLE) method.

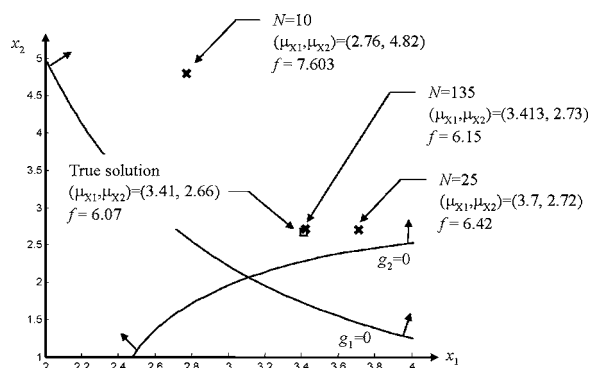
Figure 11 shows the optima obtained by fitting a distribution to the  $X_1$  samples and solving the probabilistic problem. In this figure, the cross marks are the optima obtained by fitting a distribution to the  $N$  samples, and the square is the true optimal-reliable point.

We see in this figure that for  $N=10$  and  $N=25$ , the optima are far from the true optimum. For  $N=135$ , the optimum obtained is close to the true solution but is not quite the same. An important observation about this MLE approach is that it does not provide a measure of how good the solutions are relative to the number of samples available. It is not trivial to propagate the 95% confidence intervals in Table 1 to calculate the error in the optimal solutions. Even if it is possible to do so, we will not know if the error is due to lack of data or due to improper choice of function to fit the data. The Bayesian approach avoids these difficulties, although it does require more computational effort compared to the MLE approach.

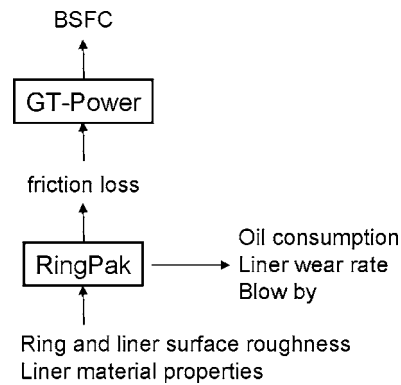
We will demonstrate the use of our reliability-based optimization method in a more complex engineering design problem next.

### 5 Case Study: Piston-Ring/Cylinder-Liner Design

The piston-ring/cylinder-liner assembly houses the combustion process inside an engine. Regardless of the engine type (spark ignition or compression ignition), this assembly must provide a tightly sealed compartment during the various stages of the engine cycle, and must efficiently transmit the force of combustion to perform mechanical work. While a tight sealing is essential to



**Fig. 11 Optima obtained by fitting a Gaussian distribution to  $X_1$  samples**



**Fig. 12 The piston-ring/cylinder-liner model**

reduce blow-by and prevent oil contamination, a certain degree of looseness is also required to permit the piston-ring pack to slide along the liner surface. The main advantage of a separate liner is that it permits reconditioning of the worn surfaces. In the absence of a liner, the entire cylinder block would have to be replaced eventually.

Both ring and liner are generally comprised of tailored materials that are manufactured with controlled surface properties. While liners have traditionally been manufactured using cast iron, more advanced materials have recently been developed that provide better weight-to-strength ratio and high wear resistance under elevated temperature. Such materials include aluminum and metal matrix composites. Surface treatments, such as electroplating, have also been used to improve contact resistance between the piston ring and the cylinder liner.

After casting and machining to near net shape, the liner surface is both rough and plateau honed to impart a deliberate amount of roughness. This roughness is considered critical for oil retention and lubrication. A comprehensive description of a liner or ring surface roughness profile requires many parameters, but among these the root mean square of asperity height is most commonly used as a single representative value to describe the surface condition.

Uncertainties in the ring and liner surface roughness directly impact the friction loss incurred during combustion, and this loss influences the performance of the engine. The case study here follows a previous one in [29] and the goal is to design an optimal-reliable ring-liner assembly to minimize the brake-specific fuel consumption (BSFC) of an engine taking into account the surface roughness uncertainties. The engine used is a V6 gasoline engine with a pre-specified geometry, materials, assembly characteristics, and other operating conditions.

The model used to predict the impact of ring and liner surface roughness on BSFC is shown in Fig. 12. Surface roughness effects on friction loss are computed using the simulation package RingPak (Ricardo Inc, London, UK). This friction loss prediction is specific to the V6 engine used. The friction loss is then input to another simulation package GT-Power (Gamma Technology Inc., Westmond, IL) to predict BSFC. Due to the large computational requirements associated with the RingPak and GT-Power simulations, neural network-based surrogate models developed in the previous study [29] are used to predict the responses of the ring-liner assembly and the engine (BSFC, liner wear rate, oil consumption, and blow-by).

Four design factors are chosen as variables in the reliability-based optimization: mean values of the ring and liner surface roughness (random variables:  $\mu_{X_1}, \mu_{X_2}$ ), and the elastic modulus and hardness of the liner (deterministic variables:  $x_3, x_4$ ). The ring-liner assembly design is subject to three constraints: the liner wear rate, the blow-by rate, and the oil consumption rate. The mathematical formulation of the design problem is as follows:

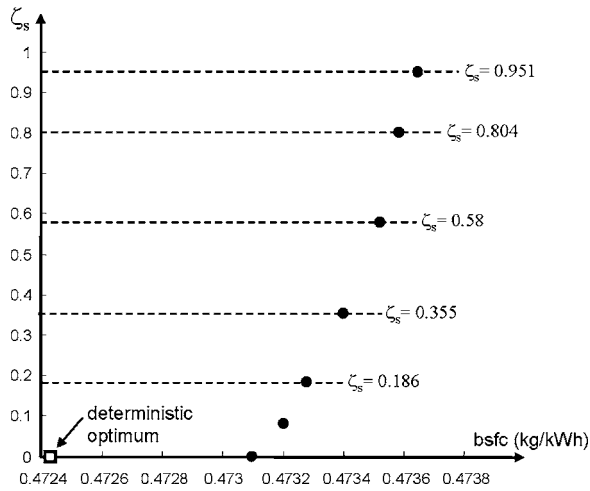


Fig. 13 Pareto optima of the ring-liner design problem

minimize BSFC

$\mu_{X_1}, \mu_{X_2}, x_3, x_4$

subject to:  $\Pr[\text{liner wear rate} \leq 2.4 \times 10^{-12} \text{ m}^3/\text{s}] \geq 0.99$

$\Pr[\text{oil consumption} \leq 15.3 \times 10^{-3} \text{ kg/h}] \geq 0.99$

$\Pr[\text{blow by} \leq 4.25 \times 10^{-5} \text{ kg/s}] \geq 0.99$

$0.1 \mu\text{m} \leq \mu_{X_1}, \mu_{X_2} \leq 1.0 \mu\text{m}$

$80 \text{ GPa} \leq x_3 \leq 340 \text{ GPa}$

$150 \text{ BHV} \leq x_4 \leq 240 \text{ BHV}$  (18)

Information on surface roughness uncertainty is incomplete in that their pdf's are not known; only a finite number of laboratory-measured samples are available. For this case study, a  $\zeta_s^{\max}$  of 0.95 is desired for the Pareto optima. From the  $N$ - $R$ - $\zeta_s^{\max}$  diagram, it is calculated that at least  $N=298$  samples of the surface roughness are needed. Three hundred actual measurements are taken and used in this study. A Bayesian reliability estimate is used to predict the three probabilistic constraints, and the problem is solved as a two-objective optimization problem with BSFC and  $\zeta_s$  as objectives. Figure 13 shows the Pareto optima obtained in the objective space.

The black circles in the figure are the Pareto optima, while the white square is the deterministic optimum of the problem without uncertainty consideration. The Pareto frontier in this case study is discrete because only samples of the surface roughness uncertainty are known (unlike in the earlier mathematical example where one of the random variables' pdf is known). As shown, to achieve 0.99 reliability we must sacrifice fuel economy; the Pareto optima are all to the right of the white square. Moreover, we have to further sacrifice fuel economy to be more confident about the designs' reliability. This second trade-off is due to the incomplete information about the surface roughness uncertainties. In all, to achieve 0.99 reliability with 95% confidence, fuel economy of the engine is reduced by  $\sim 0.25\%$  compared to the deterministic optimum, a reasonably acceptable trade-off.

Table 3 shows the  $\mu_{X_1}$ ,  $\mu_{X_2}$  and  $x_3$  values of the Pareto optima; a separate study shows that  $x_4$  has negligible impact on BSFC and the three design constraints. For comparison, variable values of the deterministic optimum are also shown. We see that  $\mu_{X_1}$  and  $x_3$  are the same for all optima, and they are at their lower bound values. This is because smoother ring surface and more rigid liner reduce friction between the ring and the liner, and thus lower BSFC. Minimizing BSFC alone would have driven  $\mu_{X_2}$  to its

Table 3 The deterministic and Pareto optima of the ring-liner case study

$\zeta_s$	BSFC (kg/kWh)	$\mu_{X_1}$ ( $\mu\text{m}$ )	$\mu_{X_2}$ ( $\mu\text{m}$ )	$x_3$ (GPa)
0.951	0.47364	0.1	0.644	80
0.804	0.47358	0.1	0.629	80
0.58	0.47351	0.1	0.615	80
0.355	0.4734	0.1	0.586	80
0.186	0.4733	0.1	0.557	80
0.084	0.4732	0.1	0.542	80
0.012	0.4731	0.1	0.513	80
...	0.4724 <sup>a</sup>	0.1	0.346	80

<sup>a</sup>Deterministic optimum

lower bound value as well; however, the oil consumption constraint limits the minimum liner surface roughness to 0.346  $\mu\text{m}$  (the deterministic optimum). Probabilistic consideration of the oil consumption constraint pushes  $\mu_{X_2}$  even higher to meet the reliability target with higher confidence, at the expense of marginal degradation in BSFC. The liner wear rate and the blow-by constraints are found inactive for all optima.

## 6 Summary

We have developed a method to estimate design reliability under incomplete information using Bayesian inference. With incomplete information, we can obtain only a distribution estimate of the reliability instead of a point estimate. Assuming a uniform prior, reliability is estimated by a Beta distribution whose precision increases as the number of samples increases. In reliability-based optimization, we can then only calculate the probability that a design will meet or exceed the target reliability, namely our confidence in the design's reliability. Using this confidence measure, the reliability-based optimization problem is converted to a two-objective problem whose Pareto optima reflect the trade-off between performance and confidence due to lack of information about the problem uncertainty.

The proposed method is a first step towards a novel way of thinking about design under uncertainty that allows direct evaluation of incomplete information; however, several issues remain. The assumption of a uniform prior distribution in Bayesian inference needs to be investigated further: the assumption is conservative with respect to the use of information, but it is not clear if it is also conservative with respect to the final design decision. The use of other prior distributions should be explored. Comparison between the proposed Bayesian approach and conventional approaches should be performed more extensively. The assumption of a collective sample set of the random quantities is also rather limiting. A more practical representation of the random data is needed for broader applicability. These are some issues under investigation in our ongoing research in this direction.

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