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QUANTIFICATION AND USE OF SYSTEM COUPLING IN DECOMPOSED DESIGN OPTIMIZATION PROBLEMS

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ABSTRACT

Decomposition-based optimization strategies are used to solve complex engineering design problems that might be otherwise unsolvable. Yet, the associated computational cost can be prohibitively high due to the often large number of separate optimizations needed for coordination of problem solutions. To reduce this cost one may exploit the fact that some systems may be weakly coupled and their interactions can be suspended with little loss in solution accuracy. Suspending such interactions is usually based on the analyst's experience or experimental observation. This article introduces an explicit measure of coupling strength among interconnected subproblems in a decomposed optimization problem, along with a systematic way for calculating it. The strength measure is then used to suspend weak couplings and thus improve system solution strategies, such as the model coordination method. Examples show that the resulting strategy can decrease the number of required system optimizations significantly.

NOMENCLATURE

f_i objective function associated with system i, $f_i : \mathbb{R}^{q_i} \to \mathbb{R}$
$\partial f/\partial \mathbf{x}$ gradient vector of $f(\mathbf{x})$ - a row vector
F objective function representing the supersystem objective, $F : \mathbb{R}^{N+\sum_{i=1}^{N} n_i} \to \mathbb{R}$
$\mathbf{g}_i$ inequality constraints associated with system i, $\mathbf{g}_i : \mathbb{R}^{q_i} \to \mathbb{R}^{m_i}$
$\partial \mathbf{g}/\partial \mathbf{x}$ Jacobian matrix of $\mathbf{g}$ with respect to $\mathbf{x}$; it is $m \times n$, if $\mathbf{g}$ is an m-vector and $\mathbf{x}$ is an n-vector
$\mathbf{h}_i$ equality constraints associated with system i, $\mathbf{h}_i : \mathbb{R}^{q_i} \to \mathbb{R}^{o_i}$
k (subscript only) denotes values at kth iteration
l_{ij} number of interaction variables associated with system interaction variable $\mathbf{y}_{ij}$
m_i number of inequality constraints associated with system inequality constraint $\mathbf{g}_i$
n_i number of design variables associated with system design variable $\mathbf{x}_i$
N total number of systems
o_i number of equality constraints associated with system equality constraint $\mathbf{h}_i$
q_i total number of design and interaction variables associated with system i, $q_i \triangleq \sum_{j=1}^{N}(n_j + l_{ji})$
$d\hat{\mathbf{x}}_j/d\mathbf{x}_i$ gradient of optimal solution of system j with respect to $\mathbf{x}_i$ for the optimization problem with $\mathbf{x}_i$ suspended
$\mathbb{R}^n$ n-dimensional Euclidean (real) space
$\mathbf{x}_i$ vector of design variables associated with system i, $\mathbf{x}_i \in \mathbb{R}^{n_i}$
$\mathbf{y}_{ij}$ data transfer or interaction variable vector from system i to system j where $\mathbf{y}_{ij} \in \mathbb{R}^{l_{ij}}$; $\mathbf{y}_{ii} \in \mathbb{R}^{l_{ii}}$ from system i to itself represents system simulation (analysis) models
Γ_i optimization coupling function vector associated with system design variable $\mathbf{x}_i$
$+, \times, ||.||$ matrix sum, matrix product and Euclidean norm respectively
$\triangleq$ definition

1 INTRODUCTION

Analysis and design of complex engineering problems often requires decomposing the problem into smaller subsystems in order to handle comprehension and computation difficulties. System solution is obtained through coordination of subsystem solutions. Such coordination is strongly affected by the interconnections or coupling of the subsystems. Intuitively, totally ”uncoupled” subsystems would require the simplest possible coordination and ”fully coupled” systems would gain little by decomposition and would defy coordination. The exact definition of coupling depends on the nature of the system problem at hand, and in our case this is the solution of system design optimization problems.

In the Multidisciplinary Design Optimization (MDO) community coordination and coupling information is often represented by the Design Structure Matrix (DSM) developed by Steward [1]. The DSM original assumption is that all task relations have strengths of one or zero (exist or not exist). Gebala and Eppinger [2] proposed a non-binary DSM that utilizes problem dependent information to assign numerical values to the couplings reflecting the strength of the relationship between tasks and the overall design. This approach is not readily extensible to design optimization. Wagner [3, 4] introduced the functional dependence table (FDT), also referred to as design incidence matrix, to assist in model-based decomposition of optimization problems; see also Krishnamachari [5] and Michelena and Papalambros [6]. The FDT is essentially the Jacobian matrix of problem functions and, as such, it does not contain binary values. However, partitioning the FDT requires filtering the partial derivative (or ”sensitivity” values) to a zero-one representation. Moreover, Jacobian values are different at different points in the design space so a universal coupling strength cannot be established unequivocally. Interestingly, the DSM is the adjacency matrix of the FDT.

Closer to our present approach, Sobieszczanski-Sobieski [7] investigated the effect of design variable changes on the interaction variables in an internally coupled system. He used the chain rule to relate total derivatives of system outputs with respect to system design variables to local system derivatives. These total derivatives are found by solving the resulting set of equations termed the Global Sensitivity Equations (GSE). Subsequently, English and Bloebaum [8, 9] proposed a method that utilizes the total derivative-based coupling sensitivity analysis to suspend interaction variables between systems during MDO coordination cycles. Emphasis was placed on single-level MDO approaches, such as multidisciplinary feasible (MDF) methods [9] [8]. The GSE formulations in these research efforts did not include optimality conditions in the definition of coupling, and so the design variables are assumed independent of each other. The work presented in this article augments the definition of GSE to include variable coupling implied by the need to satisfy optimality. The idea of generalizing the GSE to include optimality was first proposed by Sobieski [10] as a numerical tool in the bilevel decomposition strategy (BLISS). In contrast, the generalization of the GSE in this article is used in the context of coupling strength quantification.

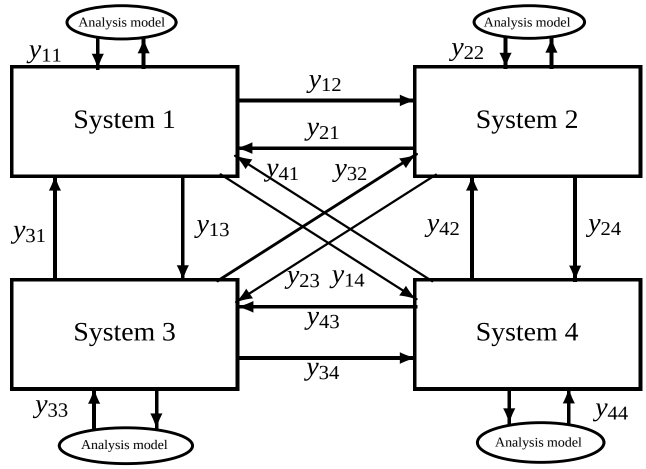

Figure 1. NON-HIERARCHICAL SYSTEM INTERACTIONS NOTATION

Several other authors have examined strength-based coupling between systems in decomposed optimization problems [11, 12]. Reyer [13] and Fathy [14] proposed the use of optimality conditions to characterize coupling. They defined coupling relative to the solution method, for example, by comparing the optimality conditions of a sequential solution strategy with the optimality conditions of the undecomposed system optimization problem. However, the effect of interaction variables (interconnections) between systems on the coupling measure was not directly considered in that work. Moreover, Fathy’s work was motivated by integration of design and control problems, and so it examined coupling between only two subsystems and assumed some objective function separability. Thus, the effect of all system variables on coupling strength was not fully explored.

This article examines how the GSEs can be modified to account for satisfaction of optimality conditions in defining subsystem coupling relevant to system optimization. This modification leads to a measure of coupling strength that can be used to suspend weak variable linking in an MDO strategy. The strategy used for demonstration is Kirsch’s model coordination method [15]. Section 2 poses the problem under consideration; Section 3 shows how the new coupling measure can be derived accounting for satisfaction of optimality; Section 4 presents the model coordination method with variable suspension strategy; and Section 5 presents an example implementation with the model coordination method. The articles concludes with a discussion of limitations and future work.

2 THE SYSTEM OPTIMIZATION PROBLEM

We consider design optimization of a supersystem that has been decomposed into several coupled systems, Fig. 1. Each system is assumed to perform its own optimization problem using its own analysis models and information from the other systems. A general non-hierarchical structure is assumed, with a hierarchical decomposition being a special case. Each system interacts (is "coupled") potentially with all the other systems though the interaction variables $\mathbf{y}_{ij} \in \mathbb{R}^{l_{ij}}$, where l_{ij} is the dimension of $\mathbf{y}_{ij}$. A direction of information flow is implied, so $\mathbf{y}_{ij}$ represents information going from system i to system j, and $\mathbf{y}_{ii}$ is used to represent information from analysis models within system i used to compute the functions (objectives or constraints) in the system optimization problem. This allows representation of various MDO schemes that involve information exchange among analyses as well as design decisions.

The design optimization problem for a supersystem with objective $F : \mathbb{R}^{N+\sum_{i=1}^{N} n_i} \rightarrow \mathbb{R}$ to be decomposed in N systems is stated as follows.

$$\begin{aligned}
&\min_{\{\mathbf{x}_i | i=1,\dots,N\}} F(f_1(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{11},\dots,\mathbf{y}_{N1}),\dots \\
&\qquad \dots, f_N(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1N},\dots,\mathbf{y}_{NN}),\mathbf{x}_1,\dots,\mathbf{x}_N) \\
&\text{subject to} \quad \mathbf{g}_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \leq \mathbf{0} \\
&\qquad\qquad \mathbf{h}_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) = \mathbf{0} \qquad (1a) \\
&\qquad\qquad i = 1,\dots,N \quad , \quad j = 1,\dots,N
\end{aligned}$$

where

$$\mathbf{y}_{ij} = Y_{ij}(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \qquad (1b)$$

represents the analysis models, namely, the functional dependence of the interaction variables on the design variables of each system $\mathbf{x}_i \in \mathbb{R}^{n_i}$ and on all other interaction variables. The vector $\mathbf{x}_i$ includes local variables specific to system i and shared variables that are common in at least two systems. Hence, the $\mathbf{x}_i$'s may have common components. The system objectives $f_i : \mathbb{R}^{q_i} \rightarrow \mathbb{R}$ and constraints $\mathbf{g}_i : \mathbb{R}^{q_i} \rightarrow \mathbb{R}^{m_i}$, $\mathbf{h}_i : \mathbb{R}^{q_i} \rightarrow \mathbb{R}^{o_i}$ may depend on other systems' design variables $\mathbf{x}_1,\dots,\mathbf{x}_N$, as well as on the interaction variables that bring information from each of the other systems. Also, n_i, m_i, o_i, l_{ij} are the dimensions of the vectors $\mathbf{x}_i$, $\mathbf{g}_i$, $\mathbf{h}_i$, $\mathbf{y}_{ij}$, respectively, and $q_i \triangleq \sum_{j=1}^{N}(n_j + l_{ji})$ is the total number of design and interaction variables associated with system i. Note that in the problem statement above no assumption is made on the form of the decomposition or the structure of the objective and constraint functions.

After decomposition, the design optimization problem for system i is stated as follows.

$$\begin{aligned}
&\min_{\mathbf{x}_i} f_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \\
&\text{subject to} \quad \mathbf{g}_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \leq \mathbf{0} \\
&\qquad\qquad \mathbf{h}_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) = \mathbf{0}. \qquad (2)
\end{aligned}$$

We assume that the optimal solution of the N system optimization problems in Eq. (2) combined with the analysis equations in Eq. (1b) will yield the supersystem optimization problem optimal solution in Eq. (1a). Convergence to the supersystem optimal solution depends on the decomposition strategy utilized.

A variable suspension strategy during a solution process will ignore the link between two systems for one or more iterations k, namely, during suspension we set

$$\mathbf{x}_{i,k+1} = \mathbf{x}_{i,k} \ , \ \mathbf{y}_{ij,k+1} = \mathbf{y}_{ij,k} \qquad (3)$$

where the index after the comma indicates iteration number.

3 GENERALIZED COUPLING STRENGTH AND THE MODIFIED GLOBAL SENSITIVITY EQUATIONS

Sobieski's GSEs [7] provide a foundation for the work presented later in this article. Following the notation of Fig. 1 consider a three-system problem with the following analysis equations

$$\mathbf{y}_{12} = Y_{12}(\mathbf{x},\mathbf{y}_{23},\mathbf{y}_{31}) \qquad (4a)$$
$$\mathbf{y}_{23} = Y_{23}(\mathbf{x},\mathbf{y}_{12},\mathbf{y}_{31}) \qquad (4b)$$
$$\mathbf{y}_{31} = Y_{31}(\mathbf{x},\mathbf{y}_{12},\mathbf{y}_{23}) \qquad (4c)$$

where $\mathbf{x}$ are the supersystem variables. In the original Sobieski derivation all system outputs are considered identical, and so $\mathbf{y}_{12} = \mathbf{y}_{13}, \mathbf{y}_{21} = \mathbf{y}_{23}, \mathbf{y}_{31} = \mathbf{y}_{32}$; further, no internal simulation/analysis is assumed, and so $\mathbf{y}_{11} = \mathbf{y}_{22} = \mathbf{y}_{33} = \mathbf{0}$.

Using the chain rule, the total derivatives of system outputs with respect to system design variables, $d\mathbf{y}_{12}/d\mathbf{x}$, $d\mathbf{y}_{23}/d\mathbf{x}$, and $d\mathbf{y}_{31}/d\mathbf{x}$, are given by the GSEs [7]

$$\begin{bmatrix} I & -\frac{\partial \mathbf{y}_{12}}{\partial \mathbf{y}_{23}} & -\frac{\partial \mathbf{y}_{12}}{\partial \mathbf{y}_{31}} \\ -\frac{\partial \mathbf{y}_{23}}{\partial \mathbf{y}_{12}} & I & -\frac{\partial \mathbf{y}_{23}}{\partial \mathbf{y}_{31}} \\ -\frac{\partial \mathbf{y}_{31}}{\partial \mathbf{y}_{12}} & -\frac{\partial \mathbf{y}_{31}}{\partial \mathbf{y}_{23}} & I \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{y}_{12}}{d\mathbf{x}} \\ \frac{d\mathbf{y}_{23}}{d\mathbf{x}} \\ \frac{d\mathbf{y}_{31}}{d\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{y}_{12}}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{y}_{23}}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{y}_{31}}{\partial \mathbf{x}} \end{bmatrix} \qquad (5)$$

The left-hand side matrix in Eq. (5) contains the partial derivatives ("sensitivities") of system outputs with respect to changes in other systems' output. The right-hand side matrix contains the partial derivatives of system outputs with respect to system design variables. These derivatives are evaluated analytically or numerically.

We now proceed to develop a modification of the GSEs to account for optimality of supersystem design. The Lagrangians of the N problems in Eq. (2) are

$$\begin{aligned}
L_i &= f_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \qquad (6) \\
&+\boldsymbol{\mu}_i^T \mathbf{g}_i(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) + \boldsymbol{\lambda}_i^T \mathbf{h}_i(\mathbf{x}_1,\dots,\mathbf{y}_{1i},\dots,\mathbf{y}_{Ni}) \\
&\qquad i = 1,\dots,N
\end{aligned}$$

where $\boldsymbol{\lambda}_i$, $\boldsymbol{\mu}_i$ are the Lagrange multipliers for the equality, inequality constraints, respectively. The first order Karush-Kuhn-

Tucker (KKT) stationarity conditions [16] are written as

$$
\begin{aligned}
&\frac{\partial f_i}{\partial \mathbf{x}_i}+\boldsymbol{\mu}_i^T\frac{\partial \mathbf{g}_i}{\partial \mathbf{x}_i}+\boldsymbol{\lambda}_i^T\frac{\partial \mathbf{h}_i}{\partial \mathbf{x}_i}+\sum_{j=1}^{N}\frac{\partial f_i}{\partial \mathbf{y}_{ji}}\frac{d\mathbf{y}_{ji}}{d\mathbf{x}_i}\\
&+\sum_{j=1}^{N}\boldsymbol{\mu}_i^T\left(\frac{\partial \mathbf{g}_i}{\partial \mathbf{y}_{ji}}\frac{d\mathbf{y}_{ji}}{d\mathbf{x}_i}\right)+\sum_{j=1}^{N}\boldsymbol{\lambda}_i^T\left(\frac{\partial \mathbf{h}_i}{\partial \mathbf{y}_{ji}}\frac{d\mathbf{y}_{ji}}{d\mathbf{x}_i}\right)=\mathbf{0}^T \qquad (7)\\
&\boldsymbol{\lambda}_i\neq\mathbf{0}\ ,\ \boldsymbol{\mu}_i\geq\mathbf{0}\ ,\ \boldsymbol{\mu}_i^T\mathbf{g}_i=0 \qquad i=1,\ldots,N
\end{aligned}
$$

The total derivatives $d\mathbf{y}_{ij}/d\mathbf{x}_i$ in Eq. (7) can be found by taking the derivatives of Eq. (1b):

$$
\begin{aligned}
&\frac{d\mathbf{y}_{jp}}{d\mathbf{x}_i}=\frac{\partial \mathbf{y}_{jp}}{\partial \mathbf{x}_i}+\frac{\partial \mathbf{y}_{jp}}{\partial \mathbf{y}_{1j}}\frac{d\mathbf{y}_{1j}}{d\mathbf{x}_i}+\ldots+\frac{\partial \mathbf{y}_{jp}}{\partial \mathbf{y}_{Nj}}\frac{d\mathbf{y}_{Nj}}{d\mathbf{x}_i}\\
&i=1,\ldots,N \quad , \quad j=1,\ldots,N \quad , \quad p=1,\ldots,N \qquad (8)
\end{aligned}
$$

collected in matrix form as

$$
\begin{bmatrix}
I & -\frac{\partial \mathbf{y}_{12}}{\partial \mathbf{y}_{21}} & .. & -\frac{\partial \mathbf{y}_{12}}{\partial \mathbf{y}_{NN}}\\
-\frac{\partial \mathbf{y}_{21}}{\partial \mathbf{y}_{12}} & I & & -\frac{\partial \mathbf{y}_{21}}{\partial \mathbf{y}_{NN}}\\
\vdots & & .. & \\
-\frac{\partial \mathbf{y}_{NN}}{\partial \mathbf{y}_{12}} & -\frac{\partial \mathbf{y}_{NN}}{\partial \mathbf{y}_{21}} & & I
\end{bmatrix}
\begin{bmatrix}
\frac{d\mathbf{y}_{12}}{d\mathbf{x}_1}\cdots\frac{d\mathbf{y}_{12}}{d\mathbf{x}_N}\\
\frac{d\mathbf{y}_{21}}{d\mathbf{x}_1}\cdots\frac{d\mathbf{y}_{21}}{d\mathbf{x}_N}\\
\vdots\\
\frac{d\mathbf{y}_{NN}}{d\mathbf{x}_1}\cdots\frac{d\mathbf{y}_{NN}}{d\mathbf{x}_N}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial\mathbf{y}_{12}}{\partial\mathbf{x}_1}\cdots\frac{\partial\mathbf{y}_{12}}{\partial\mathbf{x}_N}\\
\frac{\partial\mathbf{y}_{21}}{\partial\mathbf{x}_1}\cdots\frac{\partial\mathbf{y}_{21}}{\partial\mathbf{x}_N}\\
\vdots\\
\frac{\partial\mathbf{y}_{NN}}{\partial\mathbf{x}_1}\cdots\frac{\partial\mathbf{y}_{NN}}{\partial\mathbf{x}_N}
\end{bmatrix}
\qquad (9)
$$

which are Sobieski's GSEs. These must be extended to account for optimality and thus provide a measure of coupling strength, as we will see next.

Suppose that the link of System i with the rest of the systems is weak and the System i variables can be suspended. Then the system design variables $\mathbf{x}_i$ become parameters in the new optimization problem with $\mathbf{x}_i$ suspended. The coupling strength is defined as $dF^*(\mathbf{x}_i)/d\mathbf{x}_i$, namely, the sensitivity of the supersystem optimal objective with respect to $\mathbf{x}_i$. From the implicit function theorem the conditions in Eq. (7) can be solved for each $\mathbf{x}_l$, except for the optimality condition corresponding to suspended system i:

$$
\begin{aligned}
\hat{\mathbf{y}}_{jp}&=Y_{jp}(\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_N,\hat{\mathbf{y}}_{1j},\ldots,\hat{\mathbf{y}}_{Nj}) &(10a)\\
\hat{\mathbf{x}}_l&=X_l(\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_{l-1},\hat{\mathbf{x}}_{l+1},\ldots,\hat{\mathbf{x}}_N,\hat{\mathbf{y}}_{1l},\ldots,\hat{\mathbf{y}}_{Nl}) &(10b)\\
l&=1,\ldots,N \quad , \quad j=1,\ldots,N \quad , \quad p=1,\ldots,N \quad l\neq i
\end{aligned}
$$

Here X_l is the corresponding solution function. The analysis equations in Eq. (1b) are also rewritten with "hats": The system design variables $\hat{\mathbf{x}}_i$ and the interaction variables $\hat{\mathbf{y}}_{ij}$ will have different sensitivities from $\mathbf{x}_i$ and $\mathbf{y}_{ij}$, respectively, hence the "hat." Note that the optimality condition corresponding to system i with $\mathbf{x}_i$ suspended is not included in Eq. (10b). Moreover, the "hat" is not used on the suspended variable $\mathbf{x}_i$.

Let us now consider how this coupling strength is computed. The sensitivity of the supersystem optimal objective in Eq. (1a) with respect to $\mathbf{x}_i$ can be expressed as

$$
\begin{aligned}
\Gamma_i\triangleq\frac{dF}{d\mathbf{x}_i}&=\sum_{p=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial f_p}\frac{\partial f_p}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_i}\right)\\
&+\sum_{p=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial f_p}\frac{\partial f_p}{\partial \mathbf{y}_{jp}}\frac{d\hat{\mathbf{y}}_{jp}}{d\mathbf{x}_i}\right)+\sum_{j=1}^{N}\frac{\partial F}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_i} \qquad (11)
\end{aligned}
$$

The $1\times n_i$ vector Γ_i, which is the partial derivative of the system objective function with respect to the variables of system i, is the coupling function for system $\mathbf{x}_i$. In matrix form Eq. (11) is written as:

$$
\Gamma\triangleq
\begin{bmatrix}\Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_N\end{bmatrix}^T
=
\begin{bmatrix}
\frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial \mathbf{y}_{11}}\\
\vdots\\
\frac{\partial F}{\partial f_N}\frac{\partial f_N}{\partial \mathbf{y}_{1N}}\\
\vdots\\
\frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial \mathbf{y}_{N1}}\\
\vdots\\
\frac{\partial F}{\partial f_N}\frac{\partial f_N}{\partial \mathbf{y}_{NN}}
\end{bmatrix}^T
\begin{bmatrix}
\frac{d\hat{\mathbf{y}}_{11}}{d\mathbf{x}_1} & \frac{d\hat{\mathbf{y}}_{11}}{d\mathbf{x}_2} & \cdots & \frac{d\hat{\mathbf{y}}_{11}}{d\mathbf{x}_N}\\
\frac{d\hat{\mathbf{y}}_{12}}{d\mathbf{x}_1} & \frac{d\hat{\mathbf{y}}_{12}}{d\mathbf{x}_2} & \cdots & \frac{d\hat{\mathbf{y}}_{12}}{d\mathbf{x}_N}\\
\vdots & & & \\
\frac{d\hat{\mathbf{y}}_{NN}}{d\mathbf{x}_1} & \frac{d\hat{\mathbf{y}}_{NN}}{d\mathbf{x}_2} & \cdots & \frac{d\hat{\mathbf{y}}_{NN}}{d\mathbf{x}_N}
\end{bmatrix}
+
$$

$$
\begin{bmatrix}
\sum_{j=1}^{N}\frac{\partial F}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_1}\\
\vdots\\
\sum_{j=1}^{N}\frac{\partial F}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_N}
\end{bmatrix}^T
+
\begin{bmatrix}
\sum_{p=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial f_p}\frac{\partial f_p}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_1}\right)\\
\vdots\\
\sum_{p=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial f_p}\frac{\partial f_p}{\partial \mathbf{x}_j}\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_N}\right)
\end{bmatrix}^T
\qquad (12)
$$

where Γ is defined as the vector collection of all Γ_i's.

Given a feasible design point, most of the elements of the coupling function in Eq. (11) can be evaluated readily. The objective partial derivatives are evaluated first, analytically or numerically. The next two quantities to find are the total derivatives $d\hat{\mathbf{x}}_j/d\mathbf{x}_i$ and $d\hat{\mathbf{y}}_{ij}/d\mathbf{x}_i$. Equation (10a) can be used to determine $d\hat{\mathbf{x}}_j/d\mathbf{x}_i$, $d\hat{\mathbf{y}}_{ij}/d\mathbf{x}_i$ based on local partial derivatives as follows.

$$
\begin{aligned}
&\frac{d\hat{\mathbf{y}}_{jp}}{d\mathbf{x}_i}=\frac{\partial \hat{\mathbf{y}}_{jp}}{\partial \hat{\mathbf{x}}_1}\frac{d\hat{\mathbf{x}}_1}{d\mathbf{x}_i}+..+\frac{\partial \hat{\mathbf{y}}_{jp}}{\partial \hat{\mathbf{x}}_N}\frac{d\hat{\mathbf{x}}_N}{d\mathbf{x}_i}+\frac{\partial \hat{\mathbf{y}}_{jp}}{\partial \hat{\mathbf{y}}_{1j}}\frac{d\hat{\mathbf{y}}_{1j}}{d\mathbf{x}_i}+..+\frac{\partial \hat{\mathbf{y}}_{jp}}{\partial \hat{\mathbf{y}}_{Nj}}\frac{d\hat{\mathbf{y}}_{Nj}}{d\mathbf{x}_i}\\
&i=1,\ldots,N \quad , \quad j=1,\ldots,N \quad , \quad p=1,\ldots,N \qquad (13)
\end{aligned}
$$

Suspending System 1 and taking the derivatives of Eq. (10b) we get

$$
\begin{aligned}
\frac{d\hat{\mathbf{x}}_j}{d\mathbf{x}_i}&=\frac{\partial \hat{\mathbf{x}}_j}{\partial \hat{\mathbf{x}}_1}\frac{d\hat{\mathbf{x}}_1}{d\mathbf{x}_i}+\ldots+\frac{\partial \hat{\mathbf{x}}_j}{\partial \hat{\mathbf{x}}_N}\frac{d\hat{\mathbf{x}}_N}{d\mathbf{x}_i}\\
&+\frac{\partial \hat{\mathbf{x}}_j}{\partial \hat{\mathbf{y}}_{1j}}\frac{d\hat{\mathbf{y}}_{1j}}{d\mathbf{x}_i}+\ldots+\frac{\partial \hat{\mathbf{x}}_j}{\partial \hat{\mathbf{y}}_{Nj}}\frac{d\hat{\mathbf{y}}_{Nj}}{d\mathbf{x}_i} \qquad (14)\\
&i=2,\ldots,N \quad , \quad j=1,\ldots,N \quad , \quad p=1,\ldots,N
\end{aligned}
$$

Collecting the resulting equations in matrix form gives the modified global sensitivity equations (MGSE):

$$\begin{bmatrix} I & \cdots & -\frac{\partial \hat{\mathbf{y}}_{12}}{\partial \hat{\mathbf{y}}_{NN}} & -\frac{\partial \hat{\mathbf{y}}_{12}}{\partial \mathbf{x}_1} & \cdots & -\frac{\partial \hat{\mathbf{y}}_{12}}{\partial \mathbf{x}_N} \\ -\frac{\partial \hat{\mathbf{y}}_{21}}{\partial \hat{\mathbf{y}}_{12}} & \cdots & I & -\frac{\partial \hat{\mathbf{y}}_{21}}{\partial \mathbf{x}_1} & \cdots & -\frac{\partial \hat{\mathbf{y}}_{21}}{\partial \mathbf{x}_N} \\ \vdots & & \vdots & & \ddots & \\ -\frac{\partial \hat{\mathbf{y}}_{NN}}{\partial \hat{\mathbf{y}}_{12}} & \ddots & -\frac{\partial \hat{\mathbf{y}}_{NN}}{\partial \hat{\mathbf{y}}_{21}} & -\frac{\partial \hat{\mathbf{y}}_{NN}}{\partial \mathbf{x}_1} & \cdots & -\frac{\partial \hat{\mathbf{y}}_{NN}}{\partial \mathbf{x}_2} \\ -\frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{y}}_{12}} & \cdots & -\frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{y}}_{21}} & I & \cdots & -\frac{\partial \hat{\mathbf{x}}_2}{\partial \mathbf{x}_3} \\ \vdots & & \vdots & & & \ddots \\ -\frac{\partial \hat{\mathbf{x}}_N}{\partial \hat{\mathbf{y}}_{12}} & & -\frac{\partial \hat{\mathbf{x}}_N}{\partial \hat{\mathbf{y}}_{21}} & -\frac{\partial \hat{\mathbf{x}}_N}{\partial \mathbf{x}_2} & \cdots & I \end{bmatrix} \begin{bmatrix} \frac{d\hat{\mathbf{y}}_{12}}{d\mathbf{x}_1} \\ \vdots \\ \frac{d\hat{\mathbf{y}}_{NN}}{d\mathbf{x}_1} \\ \frac{d\hat{\mathbf{x}}_2}{d\mathbf{x}_1} \\ \vdots \\ \frac{d\hat{\mathbf{x}}_N}{d\mathbf{x}_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{\mathbf{y}}_{12}}{\partial \mathbf{x}_1} \\ \vdots \\ \frac{\partial \hat{\mathbf{y}}_{N1}}{\partial \mathbf{x}_N} \\ \frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{x}}_1} \\ \vdots \\ \frac{\partial \hat{\mathbf{x}}_N}{\partial \hat{\mathbf{x}}_1} \end{bmatrix} \quad (15)$$

The local derivatives $\partial \hat{\mathbf{y}}_{ij}/\partial \hat{\mathbf{x}}_i, \partial \hat{\mathbf{y}}_{ij}/\partial \hat{\mathbf{y}}_{ji}, \partial \hat{\mathbf{x}}_j/\partial \mathbf{x}_i, \partial \hat{\mathbf{x}}_j/\partial \mathbf{y}_{ij}$ can be computed either analytically or numerically using finite differences. The terms $\partial \hat{\mathbf{y}}_{ij}/\partial \hat{\mathbf{x}}_i, \partial \hat{\mathbf{y}}_{ij}/\partial \hat{\mathbf{y}}_{ji}$ represent local analysis derivatives, while $\partial \hat{\mathbf{x}}_j/\partial \mathbf{x}_i, \partial \hat{\mathbf{x}}_j/\partial \mathbf{y}_{ij}$ represent derivatives of the optimum with respect to parameters. We use the KKT conditions at the optimum to predict the local derivatives [17], based on the assumption that constraints at the optimum remain active as $\mathbf{x}_1$ is changed. Second order derivatives of the objective and active constraints are required, as well as the Lagrange multipliers associated with the optimum design.

The MGSEs are different from the original GSEs in that they include the optimality conditions as part of the coupled system of equations used to compute the solution sensitivity. Thus, the MGSEs account for the relationship between optimization and analysis. The key point here is that in a decomposed supersystem, the effect of one system on another may be small at non-optimal feasible points but large at the optimum, which, after all, is the point of interest.

4 THE MODEL COORDINATION METHOD WITH VARIABLE SUSPENSION

We consider now the model coordination method of Kirsch [15, 18] along with a variable suspension strategy during optimization. Suspension reduces the number of system optimizations yielding a more efficient strategy for solving the model coordination method.

Model coordination is a hierarchical two-level method that solves independent system optimization problems by fixing their coordination variable. The convergence of the model coordination method is not guaranteed [15]. However, the method remains attractive in design problems because even if convergence of the coordination is not attained, the intermediate solutions are feasible and usually represent an improvement in the objective function. Consequently, the method is also known as the feasible decomposition method.

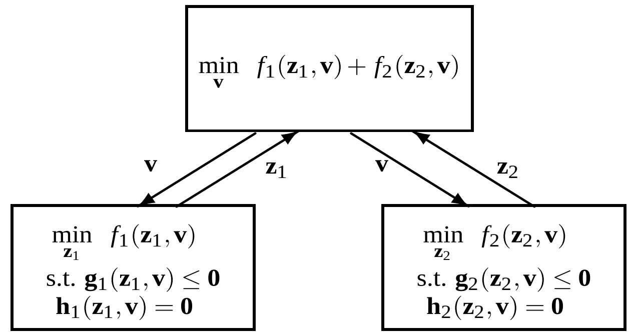

Figure 2. THE MODEL COORDINATION METHOD

Consider a supersystem decomposed using the model coordination method into several coupled systems. The undecomposed supersystem optimization problem is

$$\min_{\mathbf{z},\mathbf{v}} \; F(\mathbf{z},\mathbf{v})$$
$$\text{subject to } \; \mathbf{g}(\mathbf{z},\mathbf{v}) \leq \mathbf{0} \;, \; \mathbf{h}(\mathbf{z},\mathbf{v}) = \mathbf{0} \quad (16)$$

where $\mathbf{z} \in \mathbb{R}^n$ is the vector of design variables, and $\mathbf{v} \in \mathbb{R}^{n_3}$ is the vector of coordination variables. Let $\mathbf{z}$ be partitioned into $\mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T]^T$, $n = n_1 + n_2$, and assume that the problem and its objective function can be decomposed into the following two systems

$$F(\mathbf{z},\mathbf{v}) = f_1(\mathbf{z}_1,\mathbf{v}) + f_2(\mathbf{z}_2,\mathbf{v})$$
$$\mathbf{g}_i(\mathbf{z}_i,\mathbf{v}) \leq \mathbf{0} \;, \; \mathbf{h}_i(\mathbf{z}_i,\mathbf{v}) = \mathbf{0} \qquad i = 1,2 \quad (17)$$

where $\mathbf{z}_i \in \mathbb{R}^{n_i}$ is the vector of design variables. The model coordination method converts the supersystem optimization problem in Eq. (16) into the decomposed two-level problem in Eq. (18) and shown in Fig. 2, by fixing the coordination variable $\mathbf{v}$.

$$\text{Upper Level:} \quad \min_{\mathbf{v}} f_1(\mathbf{z}_1,\mathbf{v}) + f_2(\mathbf{z}_2,\mathbf{v})$$
$$\text{Lower Level:} \quad \min_{\mathbf{z}_i} \; f_i(\mathbf{z}_i,\mathbf{v})$$
$$\mathbf{g}_i(\mathbf{z}_i,\mathbf{v}) \leq \mathbf{0} \;, \; \mathbf{h}_i(\mathbf{z}_i,\mathbf{v}) = \mathbf{0} \qquad i = 1,2 \quad (18)$$

To characterize coupling associated with the model coordination method the problem formulation in Eq. (18) is written in terms of the coupled systems notation of Eq. (1a). Letting $\mathbf{x}_1 = \mathbf{z}_1, \mathbf{x}_2 = \mathbf{z}_2, \mathbf{x}_3 = \mathbf{v}$, Eq. (18) can be rewritten as

$$\text{SYSTEM 1:} \quad \min_{\mathbf{x}_1} \; f_1(\mathbf{x}_1,\mathbf{x}_3)$$
$$\mathbf{g}_1(\mathbf{x}_1,\mathbf{x}_3) \leq \mathbf{0} \;, \; \mathbf{h}_1(\mathbf{x}_1,\mathbf{x}_3) = \mathbf{0}$$
$$\text{SYSTEM 2:} \quad \min_{\mathbf{x}_2} \; f_2(\mathbf{x}_2,\mathbf{x}_3)$$
$$\mathbf{g}_2(\mathbf{x}_2,\mathbf{x}_3) \leq \mathbf{0} \;, \; \mathbf{h}_2(\mathbf{x}_2,\mathbf{x}_3) = \mathbf{0} \quad (19)$$
$$\text{SYSTEM 3:} \quad \min_{\mathbf{x}_3} \; f_3 = f_1(\mathbf{x}_3,\mathbf{x}_2) + f_2(\mathbf{x}_3,\mathbf{x}_3)$$

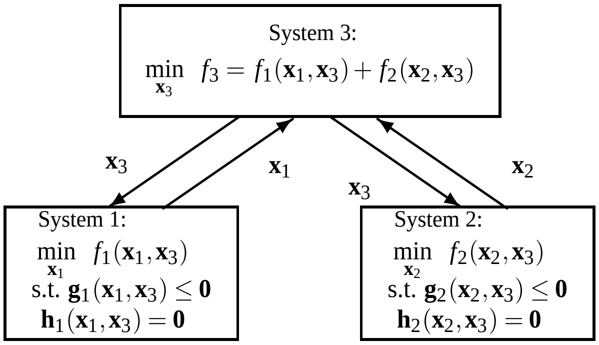

Figure 3. THE MODEL COORDINATION METHOD WRITTEN IN THE COUPLED SYSTEMS OPTIMIZATION FRAMEWORK

where System 3 represents the upper-level coordinator, Fig. 3.

Let us consider now a Hierarchical Coupling Suspension (HCS) strategy, an optimization strategy that intelligently suspends a system's optimization variables in the case of weak coupling, during at least some of the iterations. Suspending variable $\mathbf{x}_1$ of System 1 results in "systems with suspension" and an isolated system, as shown in Fig. 4.

The HCS strategy flowchart is shown in Fig. 5. The algorithm estimates $d\mathbf{x}_2^*(\mathbf{x}_1)/d\mathbf{x}_1, d\mathbf{x}_3^*(\mathbf{x}_1)/d\mathbf{x}_1$, which are the sensitivities of the optimal solution with respect to suspended variable $\mathbf{x}_1$. The algorithm then computes $df_3^*(\mathbf{x}_1)/d\mathbf{x}_1$, the sensitivity of System 3 optimal objective with respect to $\mathbf{x}_1$, which indicates coupling strength. This strength is used to determine whether to continue to suspend $\mathbf{x}_1$. After suspension, system design variable changes resulting from the optimization process alter the sensitivities $d\mathbf{x}_2^*(\mathbf{x}_1)/d\mathbf{x}_1, d\mathbf{x}_3^*(\mathbf{x}_1)/d\mathbf{x}_1, df_3^*(\mathbf{x}_1)/d\mathbf{x}_1$ and require computing new sensitivities. A trust region criterion can be used to avoid frequent updating of derivatives. After convergence, the optimization problem is solved without suspension to validate the suspension decision.

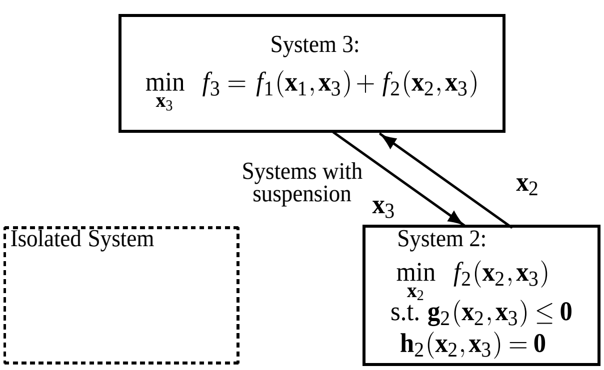

Figure 4. THE MODEL COORDINATION METHOD WITH VARIABLE $\mathbf{x}_1$ SUSPENDED

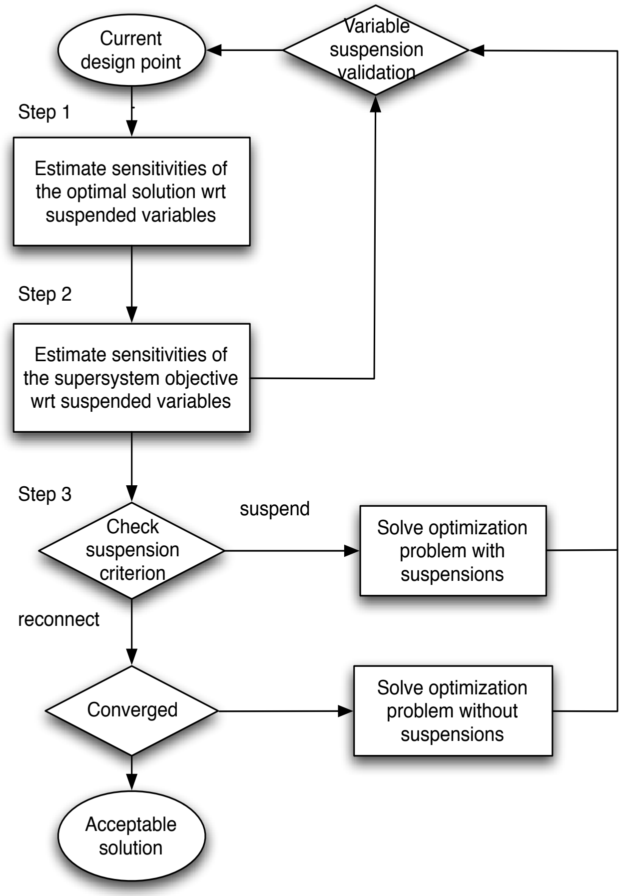

Figure 5. THE HCS ALGORITHM FLOWCHART

The algorithm's steps are described in more detail as follows.

Step 0: Initialize

Set $k = 0$ with an initial feasible design $\mathbf{x}_{1,0}, \mathbf{x}_{2,0}, \mathbf{x}_{3,0}$. Set $k = 1$ and optimize Systems 1 and 2 to yield $\mathbf{x}_{1,1}^*$ and $\mathbf{x}_{2,1}^*$ respectively; complete the iteration by utilizing $\mathbf{x}_{1,1}^*$ and $\mathbf{x}_{2,1}^*$ to optimize System 3.

Step 1: Estimate sensitivities of the optimal solution with respect to suspended variables

The sensitivities $d\mathbf{x}_2^*(\mathbf{x}_1)/d\mathbf{x}_1, d\mathbf{x}_3^*(\mathbf{x}_1)/d\mathbf{x}_1$ are needed to determine the coupling strength $df_3^*(\mathbf{x}_1)/d\mathbf{x}_1$. Note that $\mathbf{x}_1$ is a parameter in the system with suspension, so these sensitivities are parameter sensitivities at the optimal solution of Systems 2 and 3. To this end, represent Systems 2 and 3 in the format of Eq. (10b)

$$\text{(System 2)} \qquad \hat{\mathbf{x}}_2 = X_2(\mathbf{x}_1, \hat{\mathbf{x}}_3) \tag{20}$$

$$\text{(System 3)} \qquad \hat{\mathbf{x}}_3 = X_3(\mathbf{x}_1, \hat{\mathbf{x}}_2) \tag{21}$$

where the X_i's are the corresponding solution functions. The system design variable $\hat{\mathbf{x}}_i$ will have different sensitivities from $\mathbf{x}_i$, hence the "hat." The derivatives of $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ with respect to $\mathbf{x}_1$ in Eq. (20) and (21) can be expressed from Eq. (15) as

$$\begin{bmatrix} I & -\frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{x}}_3} \\ -\frac{\partial \hat{\mathbf{x}}_3}{\partial \hat{\mathbf{x}}_2} & I \end{bmatrix} \begin{bmatrix} \frac{d\hat{\mathbf{x}}_2}{d\mathbf{x}_1} \\ \frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{\mathbf{x}}_2}{\partial \mathbf{x}_1} \\ \frac{\partial \hat{\mathbf{x}}_3}{\partial \mathbf{x}_1} \end{bmatrix} \quad (22)$$

The optimal solution sensitivities $d\mathbf{x}_2^*(\mathbf{x}_1)/d\mathbf{x}_1, d\mathbf{x}_3^*(\mathbf{x}_1)/d\mathbf{x}_1$ can be determined by solving the linear equations in Eq. (22) given the local derivatives $\partial\hat{\mathbf{x}}_2/\partial\hat{\mathbf{x}}_3, \partial\hat{\mathbf{x}}_3/\partial\hat{\mathbf{x}}_2, \partial\hat{\mathbf{x}}_2/\partial\mathbf{x}_1, \partial\hat{\mathbf{x}}_3/\partial\mathbf{x}_1$. The difficulty is that $d\hat{\mathbf{x}}_2/d\mathbf{x}_1$ and $d\hat{\mathbf{x}}_3/d\mathbf{x}_1$ have to be calculated at the optimal solution of the problem with $\mathbf{x}_1$ suspended to satisfy the functional relationship they were derived from. This requirement does not provide a criterion for suspension, only a check at the final iteration to see if the suspension was correct.

Here, Eq. (22) is evaluated at the current feasible point and the values of $d\hat{\mathbf{x}}_2/d\mathbf{x}_1$ and $d\hat{\mathbf{x}}_3/d\mathbf{x}_1$ are only estimates of the sensitivities at the optimum.

Step 2: Estimate sensitivities of the supersystem objective with respect to suspended variables

The suspension decision depends on the value of $df_3^*(\mathbf{x}_1)/d\mathbf{x}_1$, which indicates coupling strength. To explain the meaning of the suspension decision associated with $df_3^*(\mathbf{x}_1)/d\mathbf{x}_1$, assume a first-order Taylor series expansion of the System 3 objective function at the optimum with suspension

$$\partial f_3^* = \frac{\partial f_3^*}{\partial \mathbf{x}_1}\partial\mathbf{x}_1 + \frac{\partial f_3^*}{\partial \mathbf{x}_2}\partial\mathbf{x}_2 + \frac{\partial f_3^*}{\partial \mathbf{x}_3}\partial\mathbf{x}_3 \quad (23)$$

The optimal solution sensitivities $d\hat{\mathbf{x}}_2(\mathbf{x}_1)/d\mathbf{x}_1, d\hat{\mathbf{x}}_3(\mathbf{x}_1)/d\mathbf{x}_1$ can be used to relate $\partial\mathbf{x}_1$ to $\partial\mathbf{x}_2$ and $\partial\mathbf{x}_3$ at the optimum with $\mathbf{x}_1$ suspended

$$\partial\mathbf{x}_2 = \frac{d\hat{\mathbf{x}}_2}{d\mathbf{x}_1}\partial\mathbf{x}_1 \quad , \quad \partial\mathbf{x}_3 = \frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_1}\partial\mathbf{x}_1. \quad (24)$$

Then Eq. (23) gives

$$\partial f_3^* = \Gamma_1\partial\mathbf{x}_1 = \left(\frac{\partial f_3^*}{\partial \mathbf{x}_1} + \frac{\partial f_3^*}{\partial \mathbf{x}_2}\frac{d\hat{\mathbf{x}}_2}{d\mathbf{x}_1} + \frac{\partial f_3^*}{\partial \mathbf{x}_3}\frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_1}\right)\partial\mathbf{x}_1 \quad (25)$$

where Γ_1 is the coupling function defined from Eq. (11) as

$$\Gamma_1 \triangleq \frac{df_3^*(\mathbf{x}_1)}{d\mathbf{x}_1} = \frac{\partial f_3^*}{\partial \mathbf{x}_1} + \frac{\partial f_3^*}{\partial \mathbf{x}_2}\frac{d\hat{\mathbf{x}}_2}{d\mathbf{x}_1} + \frac{\partial f_3^*}{\partial \mathbf{x}_3}\frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_1} \quad (26)$$

The coupling function Γ_1 represents the effect of a perturbation $\partial\mathbf{x}_1$ on the optimal objective function value of System 3 with suspension. If Γ_1 is "small" then one can assume weak coupling and suspend variable $\mathbf{x}_1$.

Similarly, if $\mathbf{x}_2$ is suspended we get

$$\partial f_3^* = \Gamma_2\partial\mathbf{x}_2 = \left(\frac{\partial f_3^*}{\partial \mathbf{x}_1}\frac{d\hat{\mathbf{x}}_1}{d\mathbf{x}_2} + \frac{\partial f_3^*}{\partial \mathbf{x}_2} + \frac{\partial f_3^*}{\partial \mathbf{x}_3}\frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_2}\right)\partial\mathbf{x}_2 \quad (27)$$

$$\Gamma_2 \triangleq \frac{df_3^*(\mathbf{x}_2)}{d\mathbf{x}_2} = \frac{\partial f_3^*}{\partial \mathbf{x}_1}\frac{d\hat{\mathbf{x}}_1}{d\mathbf{x}_2} + \frac{\partial f_3^*}{\partial \mathbf{x}_2} + \frac{\partial f_3^*}{\partial \mathbf{x}_3}\frac{d\hat{\mathbf{x}}_3}{d\mathbf{x}_2}. \quad (28)$$

If Γ_2 is small then one can suspend variable $\mathbf{x}_2$.

Step 3: Suspension criterion

The suspension criterion uses the relative magnitude of the Γ_i's. Here, if $||\Gamma_1|| < c||\Gamma_2||$ then suspend $\mathbf{x}_1$, if $||\Gamma_2|| < c||\Gamma_1||$ then suspend $\mathbf{x}_2$. The coupling parameter $c >> 1$ is chosen based on the designer's experience. If Γ_2 is "much larger" than Γ_1 then $\mathbf{x}_1$ can be suspended because it will have relatively little effect on ∂f_3^*. Similarly, if Γ_1 is much larger than Γ_2 then $\mathbf{x}_2$ can be suspended.

Step 4: Suspension Validation

After isolating System 1, the design variables of Systems 2 and 3 change during optimization with a corresponding change $\partial\mathbf{x}_1^*$ in the estimated $\mathbf{x}_1^*$. A large change in $\mathbf{x}_1^*$ can cause large changes in $d\hat{\mathbf{x}}_2/d\mathbf{x}_1$ and $d\hat{\mathbf{x}}_3/d\mathbf{x}_1$, making their prediction invalid. If $||\partial\mathbf{x}_1^*|| < \delta$ then estimates are considered valid. The parameter $\delta > 0$ is defined as the radius of the trust region where the linear approximation is considered acceptable. Note that this criterion does not take into consideration other design variables' effect. If $||\partial\mathbf{x}_1|| > \delta$ then the estimates are considered invalid and $d\hat{\mathbf{x}}_2/d\mathbf{x}_1$, $d\hat{\mathbf{x}}_3/d\mathbf{x}_1$ and Γ_1 must be updated. This update requires performing one System 1 optimization and recomputing $d\hat{\mathbf{x}}_2/d\mathbf{x}_1$, $d\hat{\mathbf{x}}_3/d\mathbf{x}_1$ and Γ_1.

Step 5: Reconnecting the suspended system

If any condition in Steps 3 or 4 is violated, then the isolated system must be reconnected.

Step 6: Termination rules

The algorithm is terminated typically if $||\mathbf{x}_{3,k-1} - \mathbf{x}_{3,k}|| < \varepsilon$.

The advantage of using the HCS strategy in model coordination is the expected computational savings associated with variable suspension. The computational tradeoff is between reduced system optimization runs and computation of sensitivities.

5 SUSPENSION STRATEGY EXAMPLE

The following example is a simple unconstrained optimization problem which is sufficient to illustrate the key ideas of the HCS strategy and the procedural approach for coupling calculation. The example focuses on two main ideas. The first is to demonstrate the HCS strategy on a weakly coupled system. The second is to show the computational savings of the HCS strategy when applied to the model coordination method.

Consider the problem

$$\min_{x_1,x_2,x_3} \quad f_3 = 0.2(0.3x_1 - 4 - x_3)^2 + 20(x_2 - 20 + x_3)^2 + 3x_3^2 \quad (29)$$

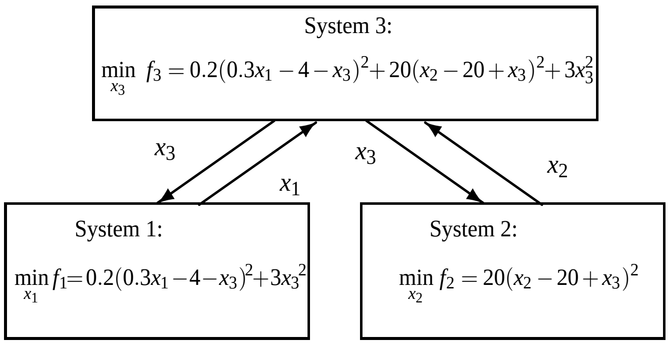

Figure 6. THE MODEL COORDINATION METHOD FOR THE UNCONSTRAINED OPTIMIZATION EXAMPLE

The stationarity conditions are

$$\frac{\partial f_3}{\partial x_1} = 0.4(0.3x_1 - 4 - x_3)0.3 = 0, \frac{\partial f_3}{\partial x_2} = 40(x_2 - 20 + x_3) = 0$$
$$\frac{\partial f_3}{\partial x_3} = -2(0.2)(0.3x_1 - 4 - x_3) + 2(20)(x_2 - 20 + x_3) + 6x_3 = 0$$

and the optimal solution is $x_1^* = 4/0.3, x_2^* = 20, x_3^* = 0$. The model coordination method is used as shown in Fig. 6. To calculate the coupling function, the optimal solution sensitivities $dx_2^*(x_1)/dx_1, dx_3^*(x_1)/dx_1$ must be determined first using Eq. (22):

$$\begin{bmatrix} 1 & 1 \\ 0.862 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\hat{x}_2}{dx_1} \\ \frac{d\hat{x}_3}{dx_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0026 \end{bmatrix} \quad , \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\hat{x}_1}{dx_3} \\ \frac{d\hat{x}_2}{dx_3} \end{bmatrix} = \begin{bmatrix} 3.33 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3.333 \\ -0.0026 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\hat{x}_1}{dx_2} \\ \frac{d\hat{x}_3}{dx_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.862 \end{bmatrix} \tag{30}$$
$$\begin{bmatrix} \frac{d\hat{x}_2}{dx_1} \\ \frac{d\hat{x}_3}{dx_1} \end{bmatrix} = \begin{bmatrix} -0.019 \\ 0.019 \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{d\hat{x}_1}{dx_2} \\ \frac{d\hat{x}_3}{dx_2} \end{bmatrix} = \begin{bmatrix} -2.899 \\ -0.869 \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{d\hat{x}_1}{dx_3} \\ \frac{d\hat{x}_2}{dx_3} \end{bmatrix} = \begin{bmatrix} 3.33 \\ -1 \end{bmatrix}$$

The coupling functions for Systems 1 and 2 are determined from Eq. (26) as

$$\Gamma_1 = \frac{\partial f_3}{\partial x_1} + \frac{\partial f_3}{\partial x_2}\frac{d\hat{x}_2}{dx_1} + \frac{\partial f_3}{\partial x_3}\frac{d\hat{x}_3}{dx_1} \; , \; \Gamma_2 = \frac{\partial f_3}{\partial x_1}\frac{d\hat{x}_1}{dx_1} + \frac{\partial f_3}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\frac{d\hat{x}_3}{dx_1} \tag{31}$$

Applying the HCS strategy, set the initial design $x_{1,0} = 11.619$, $x_{2,0} = 12.381$ and optimize System 3 to get $x_{3,0} = 6.5637$. The coupling functions in the first iteration are $\Gamma_1 = 0.8446$, $\Gamma_2 = 40.4$. Hence, System 1 is isolated and x_1 is suspended. The remaining Systems 2 and 3 are optimized until termination. Figure 7 compares iterations with and without suspension. Notice that suspension of x_1 has very little effect on the optimal solution of the overall system. As a result, ignoring System 1 reduces the computational time by a third and yields the same optimal solution. Figure 8 compares the coupling functions for the no suspension case versus the iteration steps. Γ_1 is very small compared to Γ_2 at the initial iterations, so suspending x_1 has little effect on the iteration process.

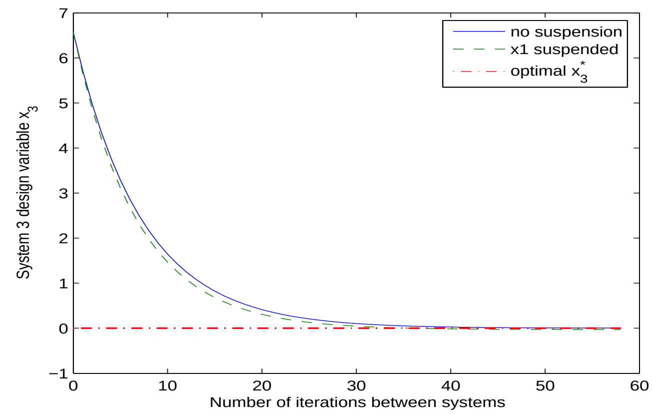

Figure 7. SYSTEM ITERATIONS WITH AND WITHOUT SUSPENSION

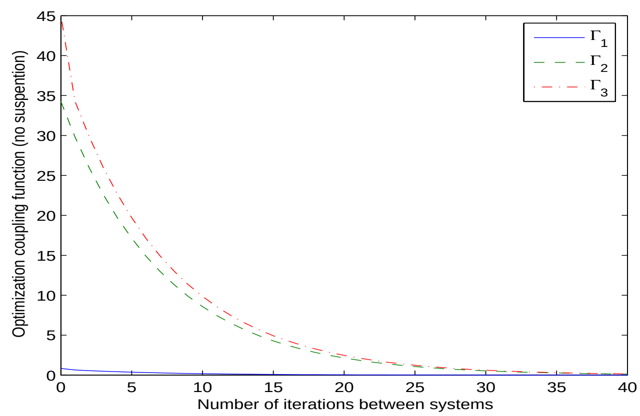

Figure 8. OPTIMIZATION COUPLING FUNCTION BEHAVIOR WITHOUT SUSPENSION

The HCS strategy demonstrated considerable computational efficiency for the model coordination method by suspending System 1. The supersystem problem required 60 iterations and the solution of 180 system optimization problems. With System 1 suspended, the problem required 60 iterations but only 120 system optimization solutions. The savings gained by ignoring System 1 for a limited time are larger than the computational burden of solving Eq. (26).

6 CONCLUSION

This article introduced a coupling strength measure for a general non-hierarchical decomposed design optimization problem. The coupling strength measure accounts for optimality by including the optimality conditions of the decomposed supersystem along with the analysis equations in a modified form of the global sensitivity equations.

Numerical computation of the coupling function involve solving a set of linear equations that requires first and second order derivative information of the objectives, constraints and analysis equations. First-order information can be readily available from the individual system optimization problem, but obtaining second-order information can be very expensive. Future work must consider methods to deal with this cost. In addition, the coupling function depends on approximate Lagrange multipliers computed under the assumption that activity does not change. Some sort of active set strategy must be introduced to address the activity assumptions.

The Hierarchical Coupling Suspension strategy has been shown to be promising in conjunction with the model coordination method. Future work should explore HCS for MDO and multilevel algorithms, like collaborative optimization [19] and analytical target cascading [20]. Substantial numerical testing remains to be done for problems with increased complexity as well as with high function evaluation costs. The impact of problem scaling must be also investigated. However, the ability to isolate rigorously system elements with weak coupling to the overall supersystem remains attractive.

7 ACKNOWLEDGMENT

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