

# Reliability Optimization With Mixed Continuous-Discrete Random Variables and Parameters

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*Engineering design problems frequently involve a mix of both continuous and discrete random variables and parameters. However, most methods in the literature deal with only the continuous or the discrete type, but not both. In particular, no method has yet addressed problems for which the random components (variables and/or parameters) are categorically discrete. This paper develops an efficient optimization method for problems involving mixed continuous-discrete random variables and parameters. The method reduces the number of function evaluations performed by systematically filtering the discrete combinations used for estimating reliability based on their importance. This importance is assessed using the spatial distance from the feasible boundary and the probability of the discrete components. The method is demonstrated in examples and is shown to be very efficient with only small errors. [DOI: 10.1115/1.2406085]*

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## 1 Introduction

Engineering design problems frequently involve a mix of both continuous and discrete variables and parameters that are random. Due to this inherent randomness, characteristics of a design may differ significantly from their nominal values. As such, when optimizing the design it is important to ensure that it is feasible regardless of the randomness. An optimization approach that accounts for design feasibility under random condition is commonly referred to as reliability based design optimization (RBDO). The term feasibility robust optimization is also used.

Randomness in the continuous components (variables or parameters) of a design may be due to variations in the design's geometry, material properties, or operating environment. For instance, yield strength of steel is typically random. The discrete components of a design may also be random for numerous reasons: changes in lifetime operation or manufacturing error. For example, some of the teeth in a gear may wear out and break within its design lifetime. Similarly, during the operation of a fuel cell system (FCS) some of the cells may fail for various reasons, and the number of failed cells is discrete and random. The gear and FCS examples involve the so-called "categorically discrete" random components because values between the discrete jumps are not defined (e.g., there is no gear with 8.3 teeth). A discrete random component can also result from limiting a continuous random component to a set of values. For example, thickness of a sheet metal is typically discrete due to manufacturing practices.

Research in RBDO focuses almost entirely on problems with continuous random components. The two most popular approaches in continuous RBDO are the reliability index approach (RIA) and its inverse, the performance measure approach (PMA) [1–6]. These methods use the first-order reliability method

(FORM) and the second-order reliability method (SORM) to calculate the reliability of a design [7–10]. RIA and PMA are both nested-loop algorithms and as such not very efficient computationally. Many methods have been developed to improve their efficiency including the safety-factor approach (SFA) [11], the most probable point (MPP) importance sampling method [12], the sequential optimization and reliability assessment (SORA) method [13], the single-loop method [14], and many others. A comparative study of SFA and SORA is given in Ref. [15]. There is also considerable research in the moment matching methods, also known as the first-order second moment (FOSM) approach. As the name indicates, these methods use first and second order derivatives to estimate the mean and variance of the propagated uncertainty, respectively [16–18]. There are also continuous RBDO methods that do not directly fall into the RIA/PMA or FOSM categories [19–21].

Some work in RBDO for discrete random components is based on Taguchi's methodology in robust design [22,23]. However, as of the writing of this paper, methods for RBDO with mixed continuous-discrete random components are virtually nonexistent. There is only one paper in the literature dealing with this type of RBDO problem for optimization of structures [24]. In this work, the authors convert the discrete part of the random components into continuous ones via addition of equivalent constraints, and use continuous RBDO techniques to solve the problem. This approach is not applicable to problems whose discrete random components are categorical (the number of gear teeth for instance). Also, reliability values calculated by this approach are somewhat ambiguous because the calculation is performed for a continuous probability distribution that does not necessarily reflect the discrete probabilities. There is a related work in RBDO for a mix of random and interval components [25], but the method cannot be extended to a mix of continuous-discrete random components.

The objective of this article is to develop an efficient RBDO method for problems involving a mix of continuous-discrete random variables and parameters. We use a probabilistic approach and assume that the relevant probabilities are known, and that the random components are independent. The work focuses strictly on design reliability in terms of the constraints of the optimization

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problem, and objective robustness is not considered. The rest of the paper is organized as follows. Section 2 provides the mathematical formulation of the mixed continuous-discrete optimization problem of interest. Section 3 presents three methods to calculate mixed continuous-discrete reliability of a design. Section 4 demonstrates the application of the method developed with two examples. The paper concludes with a short summary in Sec. 5.

## 2 Problem Formulation

Consider the deterministic mixed continuous-discrete optimization (MCDO) problem in Eq. (1)

$$\begin{aligned} & \text{minimize } f(\mathbf{x}, \mathbf{p}) \\ & \mathbf{x} = [\mathbf{x}^c, \mathbf{x}^d] \\ & \text{subject to: } g_j(\mathbf{x}, \mathbf{p}) \leq 0, \quad j = 1, \dots, J \\ & \text{where: } \mathbf{x}_L^c \leq \mathbf{x}^c \leq \mathbf{x}_U^c \\ & x_k^d \in \mathcal{Z}_k \subseteq \mathcal{R}^1, \quad k = 1, \dots, n_d \\ & \mathbf{p} = [\mathbf{p}^c, \mathbf{p}^d] \end{aligned} \quad (1)$$

In this formulation, the design variables  $\mathbf{x}$  vary during optimization, while the design parameters  $\mathbf{p}$  are fixed. The variable  $\mathbf{x}$  contains  $n_c$  continuous components  $\mathbf{x}^c = [x_1^c, \dots, x_{n_c}^c]$  and  $n_d$  discrete components  $\mathbf{x}^d = [x_1^d, \dots, x_{n_d}^d]$ . The continuous variables are bounded by  $\mathbf{x}_L^c \leq \mathbf{x}^c \leq \mathbf{x}_U^c$ , while each discrete variable  $x_k^d$ ,  $k = 1, \dots, n_d$ , takes values from a set  $\mathcal{Z}_k$  whose number of elements is  $A_k$ . Similarly, the parameter  $\mathbf{p}$  contains  $m_c$  continuous components  $\mathbf{p}^c = [p_1^c, \dots, p_{m_c}^c]$  and  $m_d$  discrete components  $\mathbf{p}^d = [p_1^d, \dots, p_{m_d}^d]$ . We assume there is no equality constraint because reliability in this context is defined only for inequality constraints.

In mixed continuous-discrete reliability optimization (MCDRO), uncertainty in the problem is modeled as randomness in variables and parameters, Eq. (2).

$$\begin{aligned} & \text{minimize } f(\boldsymbol{\mu}_X, \boldsymbol{\mu}_P) \\ & \boldsymbol{\mu}_X = [\boldsymbol{\mu}_X^c, \boldsymbol{\mu}_X^d] \\ & \text{subject to: } \Pr[g_j(\mathbf{X}, \mathbf{P}) \leq 0] \geq R_j, \quad j = 1, \dots, J \\ & \text{where: } \mathbf{X} = [\mathbf{X}^c, \mathbf{X}^d] \text{ and } \mathbf{P} = [\mathbf{P}^c, \mathbf{P}^d] \end{aligned} \quad (2)$$

For this problem, the probabilistic constraints are calculated based on the random variables  $\mathbf{X} = [\mathbf{X}^c, \mathbf{X}^d]$  and random parameters  $\mathbf{P} = [\mathbf{P}^c, \mathbf{P}^d]$ , and  $R_j$  is the desired reliability of the  $j$ th constraint specified by the designer. The probabilistic constraint can be written also as  $F_{g_j}(0) \geq R_j$ , where  $F_{g_j}$  is the cumulative distribution function (CDF) of  $g_j$ . Since we are not considering objective robustness, minimization of the objective is performed with respect to the mean values of the random variables  $\boldsymbol{\mu}_X$ , given fixed mean values of the random parameters,  $\boldsymbol{\mu}_P$ . The solution to Eq. (2) is a design that is both optimal and satisfies the reliability constraints, simply called the reliability optimum. For simplicity of discussion, we assume there is no deterministic variable or parameter.

Randomness in  $\mathbf{X}$  and  $\mathbf{P}$  is modeled as follows. Each continuous random variable  $X_i^c$ ,  $i = 1, \dots, n_c$ , is assumed randomly distributed with a known probability density function (PDF)  $f_{X_i^c}$ . Likewise, each continuous random parameter  $P_i^c$ ,  $i = 1, \dots, m_c$ , is assumed randomly distributed with a PDF  $f_{P_i^c}$ . The  $k$ th discrete random variable  $X_k^d$ ,  $k = 1, \dots, n_d$ , is assumed discretely distributed around its mean value within a set  $\mathcal{W}_k \subseteq \mathcal{Z}_k$  with a known probability mass function (PMF)  $f_{X_k^d}$ . The number of elements in  $\mathcal{W}_k$  is  $B_k \leq A_k$ . Similarly, the  $k$ th discrete random parameter  $P_k^d$ ,  $k = 1, \dots, m_d$ , is assumed discretely distributed within a set  $S_k \in \mathcal{R}^1$  with a known PMF  $f_{P_k^d}$ . The number of elements in  $S_k$  is  $C_k$ . All random variables and parameters are assumed independent.

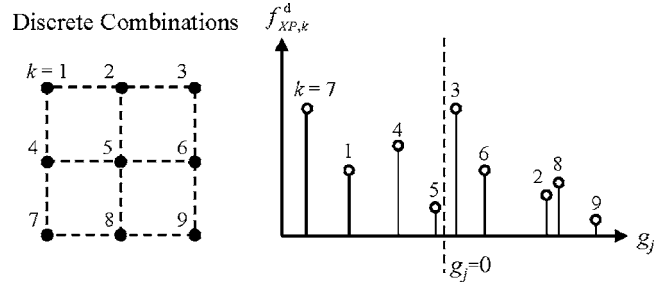


Fig. 1 Graphical representation of discrete reliability

For convenience we define two vectors  $\mathbf{Y}^c = [\mathbf{X}^c, \mathbf{P}^c]$  and  $\mathbf{Y}^d = [\mathbf{X}^d, \mathbf{P}^d]$  to be the continuous and discrete random components of the MCDRO, respectively. The number of discrete components in the problem is then  $(n_d + m_d)$ , while the number of continuous components is  $(n_c + m_c)$ . There are  $(n_d + m_d)$  discrete components, and each component ( $X_k^d$  or  $P_k^d$ ) can take a value from  $B_k$  or  $C_k$  number of choices. So the total number of combinations of all the discrete choices is  $D = (\prod_{k=1}^{n_d} B_k) (\prod_{k=1}^{m_d} C_k)$ . Since we assume independence, the joint PMF of the  $k$  discrete combination is  $f_{\mathbf{X}^d \mathbf{P}^d, k} = (\prod_{i=1}^{n_d} f_{X_i^d}) (\prod_{i=1}^{m_d} f_{P_i^d})$ , for  $k = 1, \dots, D$ . Similarly, the joint PDF of the continuous component is  $f_{\mathbf{X}^c \mathbf{P}^c} = (\prod_{i=1}^{n_c} f_{X_i^c}) (\prod_{i=1}^{m_c} f_{P_i^c})$ . For simplicity of notation, we define  $f_{\mathbf{X}^d \mathbf{P}^d, k}^d \equiv f_{\mathbf{X}^d \mathbf{P}^d, k}$  and  $f_{\mathbf{X}^c \mathbf{P}^c}^c \equiv f_{\mathbf{X}^c \mathbf{P}^c}$  to be the joint PMF and PDF, respectively.

The most important and difficult aspect of MCDRO is calculating the probabilistic constraints efficiently. Once these constraints are calculated, we can then use a conventional optimization algorithm to find the reliability optimum. The next section provides a discussion on how to calculate these constraints efficiently.

## 3 Reliability Analysis

If there is no continuous component in the problem (i.e.,  $\mathbf{Y}^c = []$ ), reliability of the  $j$ th constraint can be calculated by summing the joint probability of all feasible discrete combinations as shown in Eq. (3)

$$\begin{aligned} \Pr[g_j(\mathbf{X}^d, \mathbf{P}^d) \leq 0] &= \sum_{k=1}^D f_{\mathbf{X}^d \mathbf{P}^d, k}^d I_{j,k} \\ I_{j,k} &= \begin{cases} 1, & g_j(\mathbf{X}_k^d, \mathbf{P}_k^d) \leq 0 \\ 0, & g_j(\mathbf{X}_k^d, \mathbf{P}_k^d) > 0 \end{cases}, \quad j = 1, \dots, J \end{aligned} \quad (3)$$

Here  $I_{j,k}$  is the feasibility indicator function for the  $j$ th constraint at the  $k$ th discrete combination, and  $\mathbf{X}_k^d$  and  $\mathbf{P}_k^d$  are the  $k$ th realization of  $\mathbf{X}^d$  and  $\mathbf{P}^d$ , respectively.

Graphically,  $f_{\mathbf{X}^d \mathbf{P}^d, k}^d$  and  $I_{j,k}$  in Eq. (3) can be shown as stem plots on the scalar  $g_j$  axis with probability values for the ordinate as shown in Fig. 1. The reliability value is then the sum of probabilities of all stems to the left of (and including) the constraint boundary  $g_j = 0$ .

When there are both continuous and discrete components in the problem, the indicator function  $I_{j,k}$  is replaced by the conditional probability that  $g_j(\mathbf{X}, \mathbf{P})$  is feasible given  $[\mathbf{X}_k^d, \mathbf{P}_k^d]$ . Theoretically, this probability is calculated by solving the following integral

$$\Pr[g_j(\mathbf{X}, \mathbf{P}) \leq 0 | \mathbf{X}_k^d, \mathbf{P}_k^d] = F_{g_j, k}^c(0) = \int_{g_j \leq 0 | [\mathbf{X}_k^d, \mathbf{P}_k^d]} f_{\mathbf{X}^c \mathbf{P}^c}^c d\mathbf{X}^c d\mathbf{P}^c \quad (4)$$

The reliability of  $g_j$  is then as shown in Eq. (5)

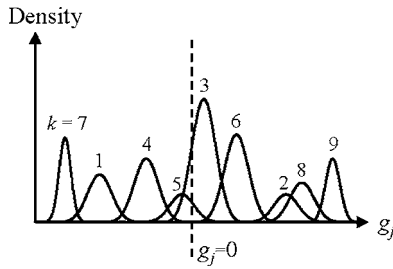


Fig. 2 Graphical representation of mixed continuous-discrete reliability

$$\Pr[g_j(\mathbf{X}, \mathbf{P}) \leq 0] = F_{g_j}(0) = \sum_{k=1}^D f_{XP,k}^d F_{g_j,k}^c(0) \quad (5)$$

Figure 2 shows the graphical representation of Eq. (5) on the  $g_j$  axis. Unlike the discrete stem plot, the ordinate in this figure is a weighted probability density. It should be emphasized that the curves shown are not PDF curves in that the area under each curve is not 1. For instance, the area under the  $k=3$  curve is clearly larger than that of the  $k=5$  curve. Rather, they are *weighted* density curves where each curve represents one multiplication element in Eq. (5). So each curve already includes the weighting factor  $f_{XP,k}^d$ . Following this definition, the reliability of  $g_j$  is the sum of the areas under the curves to the left-hand side of (and including)  $g_j=0$ .

Analytical solution to the integral in Eq. (4) is generally not possible while numerical integration is prohibitively inefficient. In the next subsections we discuss three alternative methods to calculate mixed continuous-discrete reliability: (1) Monte Carlo analysis (MCA); (2) full factorial reliability analysis (FFRA); and (3) partial factorial reliability analysis (PFRA).

**3.1 Monte Carlo Analysis.** The simplest alternative to numerical integration is to use a Monte Carlo simulation (MCS) to approximate it. There are two approaches to use MCS to calculate the reliability  $F_{g_j}(0)$ . In the first approach we use it to approximate  $F_{g_j,k}^c(0)$  only and then repeat it  $D$  times for all combinations of the discrete random components. In this approach, the continuous random components are first discretized followed by a large number of experiments. Samples for the experiments are taken proportional to the probabilities of each discretized bins;  $F_{g_j,k}^c(0)$  is then approximately equal to the ratio of feasible experiments (where  $g_j(\mathbf{X}, \mathbf{P}) \leq 0 | \mathbf{X}_k^d, \mathbf{P}_k^d$ ) to the total number of experiments. Since  $f_{XP,k}^d$  is known for all  $k$ ,  $F_{g_j}(0)$  can be calculated using Eq. (5). In the second approach we use MCS to directly approximate  $F_{g_j}(0)$ . Here, samples for the experiments are taken from both the continuous and the discrete random components. Samples for the discrete components are taken proportional to the probability mass of the discrete values.

The dimension of the first and second approach is  $(n_c + m_c)$  and  $(n_c + m_c) + (n_d + m_d)$ , respectively. In general the second approach requires more experiments for the same accuracy. However, the first approach needs to be repeated  $D$  times. Since  $D$  is much larger than  $(n_d + m_d)$ , overall the second approach is generally more efficient than the first. For the rest of the article we will use the term MCA to refer to the second approach.

MCA provides an excellent approximation to  $F_{g_j}(0)$  provided the number of experiments is large enough. In the absence of analytical solution, MCA results are often regarded as the “exact” solution. However, the large number of function evaluations needed by the method prevents its widespread use in practical applications.

**3.2 Full Factorial Reliability Analysis.** An alternative

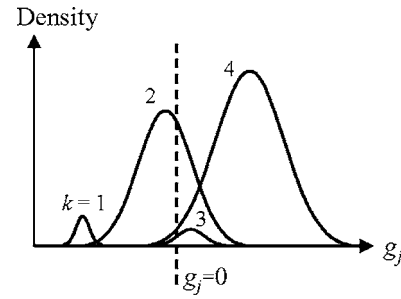


Fig. 3 Different importance of density distributions

method to calculate mixed continuous-discrete reliability is to use the FORM [8–10] to approximate the integral in Eq. (4). In this method we calculate  $F_{g_j,k}^c(0)$  using FORM at all discrete combinations, and then use them in Eq. (5).

FORM has been reported to be efficient in terms of function evaluations [8,9]. However, here we have to perform FORM  $D$  times, once for each discrete combination. Since  $D$  is generally large, the total number of function evaluations needed is still very high. If we can somehow limit the use of FORM only to those discrete combinations that really need it, we can drastically reduce the number of function evaluations required. A method to filter the discrete combinations will be discussed next.

**3.3 Partial Factorial Reliability Analysis.** In FFRA, the integral in Eq. (4) is calculated at all discrete combinations. Mathematically, however, we really only need to calculate  $F_{g_j,k}^c(0)$ 's at those discrete combinations that are close to the constraint boundary and whose  $f_{XP,k}^d/s$  are large (according to a metric that will be described later). If  $g_{j,k}$  at a  $k$ th discrete combination (with the means  $[\mu_{X^c}, \mu_{P^c}]$  for the continuous components) is far from the boundary, then  $F_{g_j,k}^c(0)$  can be approximated with  $I_{j,k}$  with a very small error. Similarly, if  $f_{XP,k}^d$  is very low,  $F_{g_j,k}^c(0)$  can also be approximated with  $I_{j,k}$  with a maximum error of only  $(0.5)f_{XP,k}^d$ .

For illustration, consider the following hypothetical weighted density distributions with  $D=4$  as shown in Fig. 3. For the  $k=1$  distribution,  $f_{XP,1}^d$  is very small and  $g_{j,1}$  is far from the boundary. So for this distribution,  $F_{g_j,1}^c(0) \approx I_{j,1} = 1.0$  with very little error. Likewise,  $f_{XP,3}^d$  is also very small and its contribution to the overall reliability is small. So although it is close to the boundary, the error induced by approximating  $F_{g_j,3}^c(0) \approx I_{j,3} = 0$  is small. In contrast, the  $k=2$  distribution is very close to the boundary and its  $f_{XP,2}^d$  is large. For this distribution  $I_{j,2} = 1.0$  is not a good approximation to  $F_{g_j,2}^c(0)$ , and we need to use FORM to calculate it more accurately. Similarly for the  $k=4$  distribution, although it is far from the boundary,  $f_{XP,4}^d$  is so large that we cannot neglect its area under the curve to the left of  $g_j=0$ . In this case too, we need to calculate  $F_{g_j,4}^c(0)$  more accurately. Notice in Fig. 3 that the areas to the left of  $g_j=0$  for density curves 3 and 4 seem comparable graphically. However,  $f_{XP,4}^d$  is larger than  $f_{XP,3}^d$  so numerically they are different (the area for density curve 4 is larger).

From the previous discussion, it is apparent that we need two pieces of information to determine if it is necessary to use FORM at the  $k$ th discrete combination: (1)  $f_{XP,k}^d$  value; and (2) relative distance of  $g_{j,k}$  from the constraint boundary. The first piece of information is already available as part of our problem formulation and assumption; the second piece of information is not as readily available.

To determine how far  $g_{j,k}$  is from the constraint boundary, we define an *influence function*  $h_{j,k}(g_{j,k}): \mathcal{R}^1 \rightarrow \mathcal{R}^1$  to be a monotonically decreasing mapping of the distance from the constraint boundary. The function is maximum when  $g_{j,k}=0$ , and gradually

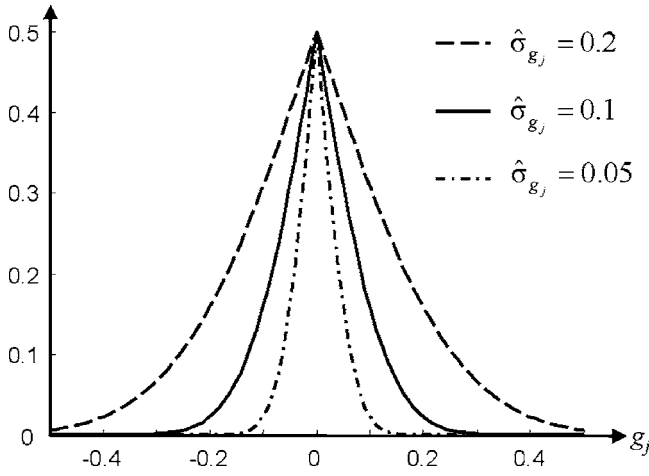


Fig. 4 Influence functions for different  $\hat{\sigma}_{g_j}$

decreases as  $g_{j,k}$  moves away from the boundary. Many types of influence function can be used, e.g., Gaussian or parabolic. In this PFRA method we use the following influence function

$$h_{j,k}(g_{j,k}) = \begin{cases} 1 - \Phi(-g_{j,k}/\hat{\sigma}_{g_j}), & \text{if } g_{j,k} \leq 0 \\ \Phi(-g_{j,k}/\hat{\sigma}_{g_j}), & \text{if } g_{j,k} > 0 \end{cases} \quad (6)$$

where  $\Phi(\cdot)$  is the CDF of a standard normal distribution with zero mean and standard deviation of 1; and  $\hat{\sigma}_{g_j}$  is a user-specified parameter. A nice feature of this influence function is that it readily provides an estimate to  $F_{g_j}^c(0)$ . If we assume the weighted conditional densities are normally distributed with a standard deviation of  $\hat{\sigma}_{g_j}$ , then

$$F_{g_j,k}^c(0) \approx \hat{F}_{g_j,k}^c = \begin{cases} 1 - h_{j,k}, & \text{if } g_{j,k} \leq 0 \\ h_{j,k}, & \text{if } g_{j,k} > 0 \end{cases} \quad (7)$$

In using Eqs. (6) and (7), we have invoked an assumption that the conditional densities are normally distributed. However, this is only one way to define an influence function. If the designer has reasons to believe that the conditional densities follow nonnormal distributions, this information may be used to construct a new influence function using the CDF of that non-normal distribution and the appropriate statistics as inputs.

The  $h_{j,k}$  function in Eq. (6) has a maximum value of 0.5 at  $g_{j,k}=0$ , and gradually decays with a distance from the boundary according to the error function  $\text{erf}(\cdot)$ . The decay rate of the function depends on the specified  $\hat{\sigma}_{g_j}$ ; the larger it is the slower the decay. Figure 4 shows the graphs of  $h_{j,k}$  for three values of  $\hat{\sigma}_{g_j}$ .

A large  $\hat{\sigma}_{g_j}$  places more importance to density curves far away from the boundary and increases the number of discrete combinations chosen for FORM analysis. This in turn will increase the total number of function evaluations. In general, however, the accuracy of the calculated reliability will also improve. In contrast, a small  $\hat{\sigma}_{g_j}$  will decrease the number of function evaluations, but generally at the expense of accuracy. In our implementation, we use the value  $\hat{\sigma}_{g_j}=0.1$ . One very important remark regarding this choice of value: since  $\hat{\sigma}_{g_j}$  is our approximation to the density curves' standard deviations, it is critical that  $g_j$  is numerically scaled to be within the same order of magnitude. In engineering problems, a typical scaling procedure is to use the upper/lower bound of the constraints as the normalizing factor.

Using  $f_{XP,k}^d$  and  $h_{j,k}$ , we can now determine if the  $k$ th discrete combination is important enough to warrant a FORM analysis. We define a quantity  $H_{j,k}(g_{j,k}) = f_{XP,k}^d h_{j,k}$  to be the *importance function* of the  $k$ th combination. If  $H_{j,k}$  is large, then it is necessary to perform FORM at this particular combination. If  $H_{j,k}$  is small,

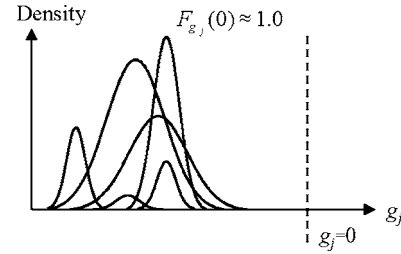


Fig. 5 Illustration of negligible importance

then we can use  $\hat{F}_{g_j,k}^c$  as an approximation.

The step-by-step algorithm of the PFRA method to calculate mixed continuous-discrete reliability of the  $j$ th constraint is as follows:

Step 1. Set  $\hat{\sigma}_{g_j}=0.1$  and a lower bound on the sum of normalized importance value  $S\bar{H}_{\min}=0.95$ .

Step 2. Calculate  $g_{j,k}$ ,  $h_{j,k}$ , and  $H_{j,k}$  for all  $k=1, \dots, D$ . Calculate the approximation  $\hat{F}_{g_j,k}^c$ .

Step 3. Calculate the sum of all importance values  $SH = \sum_{k=1}^D H_{j,k}$ .

Step 4. If  $SH \leq 0.001$ , approximate the reliability  $F_{g_j}(0) \approx \sum_{k=1}^D f_{XP,k}^d \hat{F}_{g_j,k}^c$ , then stop. Else continue.

Step 5. Calculate the normalized importance value  $\bar{H}_{j,k} = H_{j,k}/SH$  for all  $k=1, \dots, D$ .

Step 6. Sort the  $\bar{H}_{j,k}$ s from largest to smallest. Start from the largest  $\bar{H}_{j,k}$ , select  $NL \leq D$  discrete combinations such that  $\sum_{i=1}^{NL} \bar{H}_{j,i} \geq S\bar{H}_{\min}$ .

Step 7. From the selected  $NL$  combinations, discard the  $ND \leq NL$  combinations whose  $\bar{H}_{j,k} \leq 0.001$ .

Step 8. For the selected  $NP = NL - ND$  combinations, use FORM to calculate  $\hat{F}_{g_j,k}^c$ . For the rest of the combinations, keep the

$\hat{F}_{g_j,k}^c$  computed in Step 2.

Step 9. The approximate reliability of the constraint is  $F_{g_j}(0) \approx \sum_{k=1}^D f_{XP,k}^d \hat{F}_{g_j,k}^c$ . Stop.

There are two user-specified parameters in the algorithm:  $\hat{\sigma}_{g_j}$  and  $S\bar{H}_{\min}$ . The effect of  $\hat{\sigma}_{g_j}$  on the total number of function evaluations and the accuracy of the result has been discussed previously. The parameter  $S\bar{H}_{\min}$  controls the minimum percentage of total  $H_{j,k}$ 's that are selected. The larger  $S\bar{H}_{\min}$  the more discrete combinations will be selected. This will increase the accuracy of the approximation, but with an increase in the number of function evaluations. A decrease in  $S\bar{H}_{\min}$  will result in the opposite effect. In our implementation, we set  $S\bar{H}_{\min}=0.95$ , i.e., 95% of the total  $H_{j,k}$ 's are selected. The  $\hat{\sigma}_{g_j}$  and  $S\bar{H}_{\min}$  values suggested in this paper are based on our experience in applying the algorithm. They are found to result in less than 1% approximation error.

The algorithm above contains two filtering steps: *negligible importance* filtering (Step 4) and *concentrated importance* filtering (Step 7). Step 4 in the algorithm accounts for the case where all the density curves are far from the constraint boundary, as indicated by the low  $SH$  value (see Fig. 5). In this case, the  $\hat{F}_{g_j,k}^c$  approximation is sufficient for all discrete combinations, and no FORM analysis is needed at all. Step 7 in the algorithm accounts for the case where only a few of the  $NL$  discrete combinations selected in Step 6 are really important (see Fig. 6). This is indicated by the very small  $\bar{H}_{j,k}$ 's of the other  $ND$  combinations.

Note that in the above algorithm we have used a normal distri-

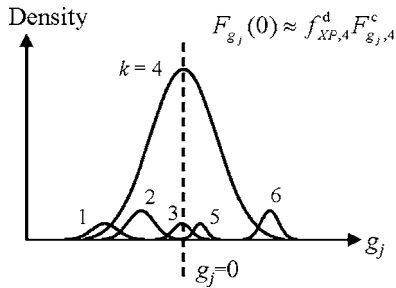


Fig. 6 Illustration of concentrated importance

bution to approximate  $F_{g_j,k}^c(0)$ . If the actual weighted conditional densities are nonsymmetric, then other forms of distributions might give better approximations.

**3.4 Error and Efficiency.** Of the three methods presented to calculate reliability, MCA is the most accurate followed by FFRA and PFRA. As mentioned before, when analytical solution is not available, MCA solution is often regarded as the true solution. If the curvature of the constraint function is not too large, the FFRA solution will also be very close to that of MCAs. Unfortunately, the FFRA error due to FORM linearization is problem dependent, and there is no analytical estimation to it. Nevertheless, the error induced by FORM has been reported to be small [8,9].

Since PFRA uses FORM, it is also affected by the linearization error. In addition to this error, PFRA also induces an error by approximating the conditional probabilities at some of the discrete combinations. Like the FORM error, the actual amount of this error is problem dependent and is not possible to formulate analytically. In the best case, all the density curves are far from the boundary (negligible importance case), and the PFRA error is zero. In the worst case, however, the normal approximation and the  $\hat{\sigma}_{g_j}$  estimate might be very different from the actual density curves. In this situation, the maximum error of the conditional probabilities not calculated by FORM is  $(0.5)f_{XP,k}^d$ . Since we select NP combinations for FORM analysis, the total maximum error is

$$e_{\max} = \sum_{k=1}^{D-NP} (0.5)f_{XP,k}^d \quad (8)$$

This error does not include the FORM linearization error. Also, Eq. (8) involves the quantity NP that is problem dependent, and so the error calculation is also problem dependent. Since we impose the  $\overline{SH}_{\min}=0.95$  selection criterion in the PFRA algorithm (Step 6), NP is generally large and the  $f_{XP,k}^d$ s at the  $(D-NP)$  combinations are small. So overall the maximum error is small.

Although MCA is the most accurate, it is also the most inefficient of the three methods in terms of number of function evaluations (FEs). The actual magnitude of MCA's FE depends on the problem's dimension, but  $FE_{MCA} > 10^6$  is typical. For the FFRA method,  $FE_{FFRA} = (D)FE_{FORM}$ , where  $FE_{FORM}$  is the number of function evaluations performed by FORM. There is no analytical formula for  $FE_{FORM}$ , but it is reported to be on the order of  $O(10^0) - O(10^2)$  [13]. Due to the efficiency of FORM, in general FFRA is much more efficient than MCA, even though  $FE_{FFRA}$  involves the factor  $D$ .

PFRA uses FORM at some, but not all, of the  $D$  discrete combinations. The FE of PFRA is  $FE_{PFRA} = D + \alpha(D)FE_{FORM}$ , where  $0 \leq \alpha \leq 1$ . The  $D$  in the first factor of the sum accounts for the evaluations of  $g_{j,k}$  at all  $k=1, \dots, D$  (Step 2). In the best case (negligible importance case),  $\alpha=0$  and  $FE_{PFRA}=D$ , i.e., no FORM analysis is needed at all. In the worst case, PFRA performs FORM at all discrete combinations. For this case,  $\alpha=1$  and  $FE_{PFRA} = D(1+FE_{FORM}) \approx FE_{FFRA}$  for  $FE_{FORM} \gg 1$ , i.e., PFRA is as effi-

Table 1  $H_1$  values for  $[X_2, P]$  combinations

	$X_2=5$	6	7	8	9
$P=2$	0.0003	0.0022	0.0108	0.0023	0.0004
$P=3$	0.007	0.0163	0.0591	0.0191	0.0052
$P=4$	0.007	0.0141	0.0582	0.0227	0.0087
$P=5$	0.0027	0.0063	0.0301	0.0127	0.0026

cient as FFRA. The quantity  $\alpha$  decreases as the design point moves away from the constraint boundary, so PFRA is more efficient for design points far from the boundary (as is usually the case in the early iterations of an optimization run). Overall, PFRA is at least as efficient as FFRA, while MCA is the least efficient of all.

A study of the error and efficiency properties of MCA, FFRA, and especially PFRA is given in the next section through examples.

## 4 Demonstration Examples

We apply the PFRA algorithm to two examples. In the first, we show a step-by-step implementation of the algorithm to calculate a mixed continuous-discrete reliability of a design point with respect to a single inequality constraint. In the second, we show the application of PFRA in an optimization algorithm to solve a MCDRO problem.

**4.1 Single Constraint Reliability.** Consider a quadratic constraint  $g_1$  that is a function of one continuous variable  $X_1$ , one integer variable  $X_2$ , and one integer parameter  $P$ . The variables and parameters in this problem are random. Following the notation in Sec. 2,  $\mathbf{X}^c=[X_1]$  and  $\mathbf{X}^d=[X_2]$ , where  $n_c=1$  and  $n_d=1$ ; for the parameters,  $\mathbf{P}^c=[]$  and  $\mathbf{P}^d=[P]$ , where  $m_c=0$  and  $m_d=1$ ; and so  $\mathbf{Y}^c=[X_1]$  and  $\mathbf{Y}^d=[X_2, P]$ . The objective of this study is to calculate the reliability of a design point whose means are  $\boldsymbol{\mu}_{\mathbf{X}} = [\mu_{X_1}, \mu_{X_2}] = [5.5, 7]$  with respect to  $g_1$ .

$$g_1(\mathbf{X}, \mathbf{P}) = \frac{1}{350}(7X_1^2 + 6X_2^2 + 8P^2 - 6X_1P + 4X_2P - 15.8X_1 - 93.2X_2 - 63P) + 1 \leq 0 \quad (9)$$

Randomness in the variables and the parameter is as follows (all are independent). The variable  $X_1$  is randomly distributed according to the normal PDF  $f_{X_1} = N(\mu_{X_1}, 0.2)$ . The integer variable  $X_2$  is distributed around its nominal value within the set  $\mathcal{W}_1 = \{5, 6, 7, 8, 9\}$  according to PMF  $f_{X_2} = \{0.05, 0.15, 0.6, 0.15, 0.05\}$ . The integer parameter  $P$  is distributed within the set  $\mathcal{S}_1 = \{2, 3, 4, 5\}$  with PMF  $f_P = \{0.05, 0.3, 0.45, 0.2\}$ . Following our notation,  $B_1=5$ ,  $C_1=4$ , and  $D=B_1C_1=20$ . The joint PMF of  $X_2$  and  $P$  is shown below ( $f_{X_2}$  as columns,  $f_P$  as rows)

$$f_{XP}^d = \begin{bmatrix} 0.0025 & 0.0075 & 0.03 & 0.0075 & 0.0025 \\ 0.015 & 0.045 & 0.18 & 0.045 & 0.015 \\ 0.0225 & 0.0675 & 0.27 & 0.0675 & 0.0225 \\ 0.01 & 0.03 & 0.12 & 0.03 & 0.01 \end{bmatrix} \quad (10)$$

The step-by-step implementation of the PFRA algorithm to calculate reliability is as follows:

Step 1. Set  $\hat{\sigma}_{g_1}=0.1$  and  $\overline{SH}_{\min}=0.95$ .

Step 2. The  $g_1$  values at all discrete combinations are calculated by substituting  $\mu_{X_1}=5.5$  and the  $[X_2, P]$  combination to Eq. (9). Using  $g_1$ , the  $h_1$  values are calculated using Eq. (6). The importance values  $H_1$  are obtained by multiplying  $h_1$  with the joint pmf  $f_{XP}^d$  in Eq. (10). The approximation  $\hat{F}_{g_1}^c$  at each combination is calculated using Eq. (7). The calculated  $H_1$  and  $\hat{F}_{g_1}^c$  are shown in Tables 1 and 2, respectively.

Step 3. The sum of the  $H_{1,k}$ s in Table 1 is  $SH = \sum_{k=1}^{20} H_{1,k}$

**Table 2**  $\hat{F}_{g_1}^c$  values for  $[X_2, P]$  combinations

	$X_2=5$	6	7	8	9
$P=2$	0.134	0.287	0.361	0.311	0.165
$P=3$	0.468	0.638	0.672	0.577	0.344
$P=4$	0.688	0.791	0.784	0.663	0.387
$P=5$	0.727	0.791	0.750	0.577	0.264

=0.2878.

Step 4. From Step 3,  $SH > 0.001$ . So this is not a negligible case; continue to Step 5.

Step 5. The normalized importance values  $\bar{H}_1$  is shown in Table 3.

Step 6. Starting from the largest  $\bar{H}_1$  in Table 3,  $NL=14$  combinations are selected such that  $\sum_{k=1}^{14} \bar{H}_{1,k} \geq S\bar{H}_{\min}$ . The combinations selected are highlighted in Table 3.

Step 7. From the 14 combinations selected, none has a  $\bar{H}_{1,k} \leq 0.001$ . So  $ND=0$ .

Step 8. For the final  $NP=NL-ND=14$  combinations selected, we use FORM to calculate  $\hat{F}_{g_1}^c$ . For the other combinations, we keep the  $\hat{F}_{g_1}^c$  values from Table 2. The revised  $\hat{F}_{g_1}^c$  values are shown in Table 4.

Step 9. The approximate reliability is  $F_{g_1}(0) \approx \sum_{k=1}^{20} f_{XP,k}^d \hat{F}_{g_1,k}^c = 0.8743$ . Stop.

As a benchmark, we calculate the reliability using the MCA method in which 1,000,000 samples are taken from the  $[X_2, P]$  discrete sets and the discretized  $X_1$  distribution. The MCA reliability value is  $F_{g_1}(0)|_{MCA} = 0.8701$ . This value is considered to be the "actual" value. The absolute error of PFRA result is  $e = 0.0042$ , less than the maximum error predicted in Eq. (8):  $e_{\max} = 0.0313$ . The relative error of PFRA is

$$\varepsilon = \frac{F_{g_1}(0)|_{PFRA} - F_{g_1}(0)|_{MCA}}{F_{g_1}(0)|_{MCA}} = 0.0048$$

less than 1%. For comparison, we also calculate the reliability using the FFRA method. The computed  $F_{g_1}(0)|_{FFRA} = 0.8695$  is very close to  $F_{g_1}(0)|_{MCA}$ . The discrepancy between the two can be attributed to FORM linearization error as well as roundoff error.

Table 5 shows a comparison of the  $F_{g_1}(0)$  values and the relative errors obtained using MCA, FFRA, and PFRA for different design points. Table 6 shows the number of function evaluations

**Table 3**  $\bar{H}_1$  values for  $[X_2, P]$  combinations

	$X_2=5$	6	7	8	9
$P=2$	0.001	0.0076	<b>0.0375</b>	0.008	0.0014
$P=3$	<b>0.0243</b>	<b>0.0566</b>	<b>0.2054</b>	<b>0.0664</b>	<b>0.0181</b>
$P=4$	<b>0.0243</b>	<b>0.049</b>	<b>0.2022</b>	<b>0.0789</b>	<b>0.0302</b>
$P=5$	0.0094	<b>0.0219</b>	<b>0.1046</b>	<b>0.0441</b>	0.009

**Table 4** Revised  $\hat{F}_{g_1}^c$  values for  $[X_2, P]$  combinations

	$X_2=5$	6	7	8	9
$P=2$	0.134	0.287	0.094	0.311	0.165
$P=3$	0.369	0.915	0.956	0.777	0.043
$P=4$	0.984	0.999	0.999	0.968	0.077
$P=5$	0.727	0.999	0.999	0.849	0.264

**Table 5** Comparisons of MCA, FFRA, and PFRA results

$(\mu_{X1}, \mu_{X2})$	MCA	FFRA	PFRA	$\varepsilon$
(1.7, 6)	1	1	0.9974	-0.0026
(5.5, 7)	0.8701	0.8695	0.8743	0.0048
(4.8, 9)	0.6704	0.6700	0.6679	-0.0037
(2.5, 3)	0.5564	0.5563	0.5580	0.0029
(1.9, 3)	0.4039	0.4046	0.4042	0.0007
(0.9, 10)	0.3390	0.3390	0.3373	-0.0050
(2.3, 2)	0.1268	0.1274	0.1267	-0.0008
(4.1, 2)	0.0788	0.0787	0.0782	-0.0076

performed by each method. The percent reduction in function evaluations from using PFRA compared to FFRA:  $\delta_{\text{red}} = (FE_{\text{FFRA}} - FE_{\text{PFRA}}) / FE_{\text{FFRA}}$  is also shown.

As seen in Table 5, PFRA provides an excellent estimate to MCA results for all ranges of reliability values. The relative error of the approximation is less than 1% for all eight design points. At the same time, using PFRA results in a 14–44% reduction in the number of function evaluations compared to using FFRA as shown in Table 6.

**4.2 Design of a Belleville Spring.** The objective of this problem is to optimize the Belleville spring shown in Fig. 7 for maximum rated load  $P_{\text{load}}$ . This example is originally formulated by Siddall [26], and is modified here to be a MCDRO problem. All probabilities are assumed independent.

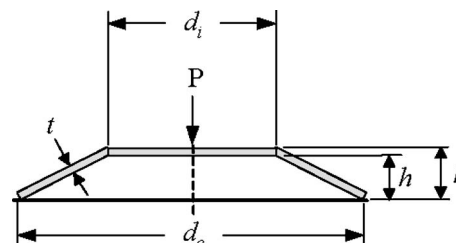
The design variables are: external diameter ( $d_e$ ), internal diameter ( $d_i$ ), free height ( $h$ ), and thickness ( $t$ ). The variables  $[d_e, d_i, h]$  are continuous (in meters), but due to manufacturing practices the sheet metal thickness ( $t$ ) is only available in multiples of 0.25 mm. There is randomness in the continuous variables  $[d_e, d_i, h]$ , and each is modeled as a normal distribution with a standard deviation of 0.0866 mm, 0.0767 mm, and 0.0333 mm, respectively [27]. The spring thickness is also random and is discretely distributed according to the following probability

$$\Pr[t = \tau] = \begin{cases} 0.5, & \tau = \mu_t \\ 0.2, & \tau = \mu_t \pm 0.25 \\ 0.05, & \tau = \mu_t \pm 0.5 \end{cases} \quad (11)$$

The spring is to be made from high strength steel Type C, but due to manufacturing imperfection its properties may vary. Table 7

**Table 6** Comparison of  $FE_{MCA}$ ,  $FE_{FFRA}$ , and  $FE_{PFRA}$

$(\mu_{X1}, \mu_{X2})$	$FE_{MCA}$	$FE_{FFRA}$	$FE_{PFRA}$	$\delta_{\text{red}}$
(1.7, 6)	1,000,000	129	111	0.140
(5.5, 7)	1,000,000	373	250	0.330
(4.8, 9)	1,000,000	279	173	0.380
(2.5, 3)	1,000,000	141	79	0.440
(1.9, 3)	1,000,000	151	115	0.238
(0.9, 10)	1,000,000	228	164	0.281
(2.3, 2)	1,000,000	112	76	0.321
(4.1, 2)	1,000,000	136	82	0.397



**Fig. 7** A Belleville spring

**Table 7 Steel properties and their probabilities**

Steel type	$E$ (GPa)	$\sigma_{aw}$ (MPa)	Probability
A	190	1100	0.1
B	200	1350	0.25
C	205	1500	0.6
D	210	1600	0.05

shows the four possible discrete variations of the steel properties, and the probabilities of each variation. For ease of referral, we assign the letter *A*, *B*, *C*, and *D* to each of the four possible variations. In this table, the quantities  $E$  and  $\sigma_{aw}$  are the elastic modulus and allowable stress of the steel, respectively.

So in this problem  $\mathbf{X}^c=[d_e, d_i, h]$  and  $\mathbf{X}^d=[t]$ , and  $n_c=3$  and  $n_d=1$ . For the parameters,  $\mathbf{P}^c=[\ ]$  and  $\mathbf{P}^d=[\text{steel type}]$ , and  $m_c=0$  and  $m_d=1$ . The continuous components are  $\mathbf{Y}^c=[d_e, d_i, h]$ , while the discrete components are  $\mathbf{Y}^d=[t, \text{steel type}]$ . The PDFs of the continuous variable are  $f_{d_e}=N(\mu_{d_e}, 0.0866)$ ,  $f_{d_i}=N(\mu_{d_i}, 0.0767)$ , and  $f_h=N(\mu_h, 0.0333)$ . For the discrete variable,  $\mathcal{V}_1=\{\mu_t-0.5, \mu_t-0.25, \mu_t, \mu_t+0.25, \mu_t+0.5\}$  with a PMF  $f_t=\{0.05, 0.2, 0.5, 0.2, 0.05\}$  and  $B_1=5$ . For the discrete parameter:  $S_1=\{A, B, C, D\}$  with a PMF  $f_{\text{steel}}=\{0.1, 0.25, 0.6, 0.05\}$  and  $C_1=4$ . The number of combinations of the discrete components is  $D=B_1 C_1=20$ .

The spring optimization is constrained by two design constraints, maximum allowable stress and maximum mass, and five geometric constraints. The formulation of the deterministic MCDO is shown below (notice how the constraints are numerically scaled)

$$\begin{aligned}
 & \text{maximize } f(\mathbf{x}, \mathbf{p}) = P_{\text{load}} \\
 & \quad \mathbf{x}=[d_e, d_i, h, t] \\
 & \text{subject to: } g_1(\mathbf{x}, \mathbf{p}) \equiv \sigma_{\text{max}}/\sigma_{\text{aw}} - 1 \leq 0 \\
 & \quad g_2(\mathbf{x}, \mathbf{p}) \equiv m/m_{\text{max}} - 1 \leq 0 \\
 & \quad g_3(\mathbf{x}, \mathbf{p}) \equiv h_{\text{min}}/h - 1 \leq 0 \\
 & \quad g_4(\mathbf{x}, \mathbf{p}) \equiv (h+t)/l - 1 \leq 0 \\
 & \quad g_5(\mathbf{x}, \mathbf{p}) \equiv d_e/d_{\text{max}} - 1 \leq 0 \\
 & \quad g_6(\mathbf{x}, \mathbf{p}) \equiv 1.25d_i/d_e - 1 \leq 0 \\
 & \quad g_7(\mathbf{x}, \mathbf{p}) \equiv 1 - 0.3(d_e - d_i)/h \leq 0 \tag{12}
 \end{aligned}$$

Here  $P_{\text{load}}$  is the rated load (N),  $\sigma_{\text{max}}$  is the maximum stress (Pa), and  $m$  is the spring mass (kg). Details of the deterministic problem can be found in Ref. [28].

The optimum of the deterministic MCDO is  $[d_e, d_i, h, t]^*=[0.3, 0.232, 5.0, 8.0]$  (using the baseline steel Type *C* as the material). The  $t^*$  and  $h^*$  values shown are in mm. The maximum load of this deterministic optimum is  $P_{\text{load}}^*=67.82$  kN. The constraint values are  $\mathbf{g}^*=[-0.0019, -0.117, 0, -0.35, 0, -0.0307, -3.038]$  where  $g_3$  and  $g_5$  are active. Based on the specified PDFs and PMFs, the reliabilities of this optimal point w.r.t. the constraints are  $\mathbf{F}_{g_j}^*(0)=[0.376, 0.994, 0.5, 1.0, 0.976, 0.982, 1.0]$ . As can be seen, the deterministic optimum has low reliabilities in terms of  $g_1$  and  $g_3$ .

**4.2.1 Reliability Optimization.** For reliability optimization, all constraints are replaced with probabilistic ones. The lower reliability bound for all probabilistic constraints is set to be  $R_j=0.99$  for  $j=1, \dots, 7$ .

We solved the MCDRO using the MOST algorithm as implemented in the commercial software iSIGHT 9.0. MOST is an

**Table 8 Comparison of MCA, FFRA, and PFRA optima**

	MCA	FFRA	PFRA
$d_e$ (m)	0.297	0.289	0.276
$d_i$ (m)	0.22	0.2065	0.168
$h$ (mm)	5.07	5.08	5.44
$t$ (mm)	6	6	6
$P_{\text{load}}$ (kN)	26.068	25.375	23.587
$F_{g_1}(0)$	0.995	0.996	0.989
$F_{g_3}(0)$	0.983	0.992	1.0
FE	4,470,000	177,752	110,866

optimization algorithm that combines SQP and branch-and-bound algorithms to solve mixed continuous-discrete problems [29]. For comparison, we solved the problem three times, each using MCA, FFRA, and PFRA for reliability calculation. For fairness, the same starting point  $[d_e, d_i, h, t]=[0.3, 0.2, 5.0, 7.0]$  is used in all three runs. For the MCA method, 1,000 samples are used (this number provides a relatively accurate reliability while still practically manageable). For the FFRA method, the HL-RF algorithm [9] is chosen for FORM calculation. For the PFRA method, the parameters are specified to be  $\hat{\sigma}_{g_j}=0.1$  and  $\bar{S}\bar{H}_{\text{min}}=0.95$ . The  $\hat{F}_{g_j}^c$  values at the selected combinations are also calculated with FORM using the HL-RF algorithm.

Table 8 shows the reliability optima obtained and the total number of function evaluations performed. In counting FE, each calculation of either the objective or constraint function is considered one evaluation. The table also shows the reliabilities of the optima in terms of  $g_1$  and  $g_3$ . Reliability values in terms of the other constraints are the same for all three optima.

We see in Table 8 that the MCA optimum has the highest  $P_{\text{load}}$  while PFRA optimum has the lowest. However, we also see that the MCA optimum does not quite satisfy the third reliability constraint. A possible explanation is that the number of samples used is not large enough so that the reliability values calculated by MCA are noisy. Even with the relatively low number of samples, however, the optimization already required more than 4 million function evaluations. This observation further demonstrates the impracticality of MCA. The FFRA's  $P_{\text{load}}$  is slightly lower than that of MCAs, but it satisfies all the reliability constraints. In terms of efficiency, the  $\text{FE}_{\text{FFRA}}$  value is also significantly lower than  $\text{FE}_{\text{MCA}}$ . PFRA's  $P_{\text{load}}$  is lower than FFRA's. This may be caused by the inconsistent changes in the reliability values (and hence not-as-accurate gradient values) calculated using PFRA. Nevertheless, in return for this 7% decrease in  $P_{\text{load}}$ , we gain a significantly larger 37% decrease in total function evaluations. These results suggest a potential hybrid optimization algorithm utilizing both PFRA and FFRA to achieve convergence and efficiency.

The PFRA method offers a way to reduce the computational requirement of solving a MCDRO problem significantly by sacrificing a little accuracy. Even with the reduction, however, the total computational cost may still be rather high. One way to further increase efficiency is to use a more efficient technique other than FORM to calculate the conditional probability. Another way is to use parallelization to calculate the conditional probability at each discrete combination. These options merit further investigation.

## 5 Summary

This paper presents a method to reduce the number of function evaluations needed to calculate a mixed continuous-discrete reliability while maintaining accuracy. Unlike the FFRA method, PFRA uses FORM to calculate the continuous conditional probabilities only at some of the discrete combinations. These discrete combinations are systematically selected based on their importance, which in turn depends on the relative distance from the

boundary and the probability of the discrete components. In the numerical example, the PFRA result is found to be in excellent agreement with the MCA value along with a significant improvement in computational efficiency. The relative error of the approximation is less than 1%, and the number of function evaluations decreases by as much as 44%. When used in reliability optimization, PFRA is also found to perform well. The PFRA optimum is slightly inferior to the FFRA optimum (7% lower), but the total number of function evaluations is decreased by 37%.

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