Proceedings of IDETC/CIE'05
2005 Int. Design Engineering Technical Conf & Computers and Information in Engineering
Long Beach, California, USA, September 24-28, 2005

DETC2005-84514

RELIABILITY OPTIMIZATION INVOLVING MIXED CONTINUOUS-DISCRETE UNCERTAINTIES

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ABSTRACT

Engineering design problems frequently involve a mix of both continuous and discrete uncertainties. However, most methods in the literature deal with either continuous or discrete uncertainties, but not both. In particular, no method has yet addressed uncertainty for categorically discrete variables or parameters. This article develops an efficient optimization method for problems involving mixed continuous-discrete uncertainties. The method reduces the number of function evaluations performed by systematically filtering the discrete factorials used for estimating reliability based on their importance. This importance is assessed using the spatial distance from the feasible boundary and the probability of the discrete components. The method is demonstrated in examples and is shown to be very efficient with only small errors.

Keywords: Filter, influence function, mixed continuous-discrete uncertainties, Reliability Based Design Optimization (RBDO)

1. INTRODUCTION

Engineering design problems frequently involve a mix of both continuous and discrete uncertainties. Because of these uncertainties, the characteristics of a design may differ significantly from its nominal values. As such, in optimizing the design it is important to ensure that the design is feasible regardless of the variations. An optimization approach that accounts for feasibility under uncertainty is commonly referred to as Reliability Based Design Optimization (RBDO). The term Feasibility Robust Optimization is also widely used.

Continuous uncertainties in the problem may come from various sources such as variations in the geometric description of the design, in its material properties, and in the operating environment. Examples of a discrete variation include variations in the number of gear teeth, in material selection, and in cross-section shape, among others. In addition to these so-called "categorical discrete" uncertainties, discrete variations could also result from limiting continuous variations to within a set of values (due to engineering practices or requirements). For example, variation in sheet metal thickness choices is typically discrete due to manufacturing practice.

In addition to physical variations, mixed continuous-discrete uncertainty is also very important in a design problem because it models the uncertainties that occur in making future projections. In forecasting, variations of the predicted target are discrete, but variations of the target itself are continuous. For example, the weight of a new composite under development may be projected to be 5%, 10%, or 20% less than what is currently available, a discrete variation. But the weight also varies due to manufacturing tolerances and variability in the material properties, a continuous variation.

Research in RBDO focuses almost entirely on continuous uncertainties. The two most popular approaches in continuous RBDO are the Reliability Index Approach (RIA) and its inverse, the Performance Measure Approach (PMA) [1–6]. These methods use the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) to calculate the reliability of a design [7–10]. RIA and PMA are both nested-loop algorithms and as such not very efficient computationally. Many methods have been developed to improve their efficiency including the Safety-Factor Approach (SFA) [11], the MPP importance sampling method [12], the Sequential Optimization and Reliability Assessment (SORA) method [13], the single-loop method

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[14], and many others. A comparative study of SFA and SORA is given in [15]. There is also considerable research in moment matching methods to calculate continuous reliability, also called the First Order Second Moment (FOSM) approach. As its name indicates, these methods use first and second order derivatives to estimate the mean and variance of the propagated uncertainties, respectively [16–18]. There are also continuous RBDO methods that do not directly fall into the RIA/PMA or FOSM categories [19–21].

Some work in RBDO for discrete uncertainties is based on Taguchi's methodology in robust design [22, 23]. Other discrete RBDO methods, such as the tolerance box approach [24, 25], have also been developed. However, as of the writing of this article, methods for mixed continuous-discrete RBDO are virtually non-existent. There is only one article in the literature dealing with mixed continuous-discrete RBDO for optimization of structures [26]. In this work, the authors convert the discrete part of the uncertainties into a continuous one via addition of equivalent constraints, and use continuous RBDO techniques to solve the problem. This approach is not applicable to problems whose discrete uncertainties are categorical (e.g., material selection), for which values between the discrete jumps are not defined. Also, reliability values calculated by this approach are somewhat ambiguous because the calculation is performed for a continuous probability distribution that does not necessarily reflect the discrete probabilities. There is a related work in RBDO for a mix of random and interval variables [27], but the method cannot be extended to a mix of continuous-discrete uncertainties.

The objective of this article is to develop an efficient RBDO method for problems involving a mix of continuous-discrete uncertainties. We use a probabilistic approach and assume that the relevant probabilities are known, and that the random variables and parameters are independent. The work focuses strictly on constraint reliability of an optimization problem, and objective robustness is not considered. The rest of the article is organized as follows. Section 2 provides the mathematical formulation of the mixed continuous-discrete optimization problem of interest. Section 3 presents three methods to calculate mixed continuous-discrete reliability of a design. Section 4 demonstrates the application of the method developed with two examples. The article concludes with a short summary in Section 5.

2. PROBLEM FORMULATION

Consider the deterministic Mixed Continuous-Discrete Optimization (MCDO) problem in Eq.(1).

$$\begin{aligned}
&\underset{\mathbf{x}=[\mathbf{x}^{\mathrm{c}},\mathbf{x}^{\mathrm{d}}]}{\text{minimize}} && f(\mathbf{x},\mathbf{p}) \\
&\text{subject to:} && g_j(\mathbf{x},\mathbf{p}) \le 0, \qquad j=1,\dots,J \\
&\text{where:} && \mathbf{x}_L^{\mathrm{c}} \le \mathbf{x}^{\mathrm{c}} \le \mathbf{x}_U^{\mathrm{c}} \\
& && x_k^{\mathrm{d}} \in Z_k \subseteq \mathbb{R}^1, \quad k=1,\dots,n_d \\
& && \mathbf{p}=[\mathbf{p}^{\mathrm{c}},\mathbf{p}^{\mathrm{d}}]
\end{aligned} \tag{1}$$

In this formulation, the design variables $\mathbf{x}$ vary during optimization, while the design parameters $\mathbf{p}$ are fixed. The variable $\mathbf{x}$ contains n_c continuous components $\mathbf{x}^{\mathrm{c}} = [x_1^{\mathrm{c}},\dots,x_{n_c}^{\mathrm{c}}]$ and n_d discrete components $\mathbf{x}^{\mathrm{d}} = [x_1^{\mathrm{d}},\dots,x_{n_d}^{\mathrm{d}}]$. The continuous variables are bounded by $\mathbf{x}_L^{\mathrm{c}} \le \mathbf{x}^{\mathrm{c}} \le \mathbf{x}_U^{\mathrm{c}}$, while each discrete variable x_k^{d}, $k=1,\dots,n_d$, takes values from a set whose number of elements is A_k. Similarly, the parameter $\mathbf{p}$ contains m_c continuous components $\mathbf{p}^{\mathrm{c}} = [p_1^{\mathrm{c}},\dots,p_{m_c}^{\mathrm{c}}]$ and m_d discrete components $\mathbf{p}^{\mathrm{d}} = [p_1^{\mathrm{d}},\dots,p_{m_d}^{\mathrm{d}}]$. We assume there are no equality constraints because constraint reliability is defined only for inequality constraints.

In Mixed Continuous-Discrete Reliability Optimization (MCDRO), uncertainties in the problem are modeled as randomness in variables and parameters, Eq.(2).

$$\begin{aligned}
&\underset{\boldsymbol{\mu}_{\mathbf{X}}=[\boldsymbol{\mu}_{\mathbf{X}^{\mathrm{c}}},\boldsymbol{\mu}_{\mathbf{X}^{\mathrm{d}}}]}{\text{minimize}} \quad f(\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}}) \\
&\text{subject to: } \Pr[g_j(\mathbf{X},\mathbf{P}) \le 0] \ge R_j, \quad j=1,\dots,J \\
&\text{where: } \mathbf{X}=[\mathbf{X}^{\mathrm{c}},\mathbf{X}^{\mathrm{d}}] \text{ and } \mathbf{P}=[\mathbf{P}^{\mathrm{c}},\mathbf{P}^{\mathrm{d}}]
\end{aligned} \tag{2}$$

For this problem, the probabilistic constraints are calculated based on the random variables $\mathbf{X}=[\mathbf{X}^{\mathrm{c}},\mathbf{X}^{\mathrm{d}}]$ and random parameters $\mathbf{P}=[\mathbf{P}^{\mathrm{c}},\mathbf{P}^{\mathrm{d}}]$, and R_j is the desired reliability of the j-th constraint specified by the designer. The probabilistic constraint can be written also as $F_{g_j}(0) \ge R_j$, where F_{g_j} is the cumulative distribution function (cdf) of g_j. Since we are not considering objective robustness, minimization of the objective is performed with respect to the mean values of the random variables $\boldsymbol{\mu}_{\mathbf{X}}$, given fixed mean values of the random parameters, $\boldsymbol{\mu}_{\mathbf{P}}$. For simplicity of discussion, we assume there is no deterministic variable or parameter.

Randomness in $\mathbf{X}$ and $\mathbf{P}$ is modeled as follows. Each continuous random variable X_i^{c}, $i=1,\dots,n_c$, is assumed randomly distributed with a known probability density function (pdf) $f_{X_i^{\mathrm{c}}}$. Likewise, each continuous random parameter P_i^{c}, $i=1,\dots,m_c$, is assumed randomly distributed with a pdf $f_{P_i^{\mathrm{c}}}$. The k-th discrete random variable X_k^{d}, $k=1,\dots,n_d$, is assumed discretely distributed around its mean value within a set $W_k \subseteq Z_k$ with a known probability mass function (pmf) $f_{X_k^{\mathrm{d}}}$. The number of elements in W_k is $B_k \le A_k$. Similarly, the k-th discrete random parameter P_k^{d}, $k=1,\dots,m_d$, is assumed discretely distributed within a set $S_k \in \mathbb{R}^1$ with a known pmf $f_{P_k^{\mathrm{d}}}$. The number of elements in S_k is C_k. All random variables and parameters are assumed independent.

For convenience we define two vectors $\mathbf{Y}^{\mathrm{c}}=[\mathbf{X}^{\mathrm{c}},\mathbf{P}^{\mathrm{c}}]$ and $\mathbf{Y}^{\mathrm{d}}=[\mathbf{X}^{\mathrm{d}},\mathbf{P}^{\mathrm{d}}]$ to be the continuous and discrete random components of the MCDRO, respectively. The number of discrete components in the problem is then (n_d+m_d), while the number of continuous components is (n_c+m_c). There are (n_d+m_d) discrete components, and each component (X_k^{d} or P_k^{d}) can take a value from B_k or C_k number of choices. So the number of full factorials of all the discrete choices is $D=(\prod_{k=1}^{n_d} B_k)(\prod_{k=1}^{m_d} C_k)$. Since we assume independence, the joint pmf of the k-th discrete factorial is $f_{\mathbf{X}^d,\mathbf{P}^d,k} = (\prod_{i=1}^{n_d} f_{X_i^d})(\prod_{i=1}^{m_d} f_{P_i^d})$, for $k = 1,\ldots,D$. Similarly, the joint pdf of the continuous component is $f_{\mathbf{X}^c,\mathbf{P}^c} = (\prod_{i=1}^{n_c} f_{X_i^c})(\prod_{i=1}^{m_c} f_{P_i^c})$. For simplicity of notation, we define $f^d_{XP,k} \equiv f_{\mathbf{X}^d,\mathbf{P}^d,k}$ and $f^c_{XP} \equiv f_{\mathbf{X}^c,\mathbf{P}^c}$ to be the joint pmf and pdf, respectively.

The most important and difficult aspect of MCDRO is calculating the probabilistic constraints efficiently. Once these constraints are calculated, we can then use a conventional optimization algorithm to find the reliability optimum. The next section provides a discussion on how to calculate these constraints efficiently.

3. RELIABILITY ANALYSIS

If there is no continuous component in the problem (i.e., $\mathbf{Y}^c = [\ ]$), reliability of the j-th constraint can be calculated by summing the joint probability of all feasible discrete factorials as shown in Eq.(3).

$$\Pr[g_j(\mathbf{X}^d,\mathbf{P}^d) \le 0] = \sum_{k=1}^{D} f^d_{XP,k} I_{j,k}$$
$$I_{j,k} = \begin{cases} 1, & \text{if } g_j(\mathbf{X}^d_k,\mathbf{P}^d_k) \le 0 \\ 0, & \text{if } g_j(\mathbf{X}^d_k,\mathbf{P}^d_k) > 0 \end{cases}, \ j = 1,\ldots,J \tag{3}$$

Where $I_{j,k}$ is the feasibility indicator function for the j-th constraint at the k-th discrete factorial, and $\mathbf{X}^d_k$ and $\mathbf{P}^d_k$ are the k-th realization of $\mathbf{X}^d$ and $\mathbf{P}^d$, respectively.

Graphically, the $f^d_{XP,k}$'s and $I_{j,k}$'s in Eq.(3) can be shown as a stem plot on the scalar g_j axis with probability values for the ordinate as shown in Fig.1. The reliability value is then the sum of probabilities of all stems to the left of and including the constraint boundary $g_j = 0$.

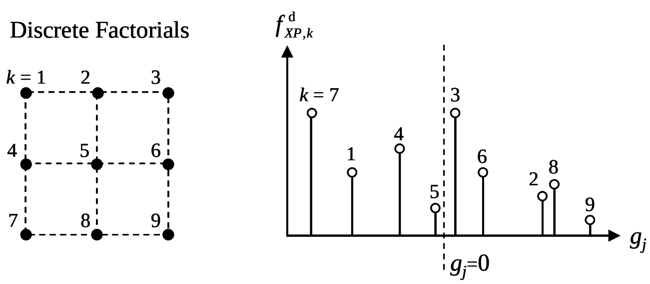

Figure 1. Graphical representation of discrete reliability

When there are both continuous and discrete components in the problem, the indicator function $I_{j,k}$ is replaced by the conditional probability that $g_j(\mathbf{X},\mathbf{P})$ is feasible given $[\mathbf{X}^d_k,\mathbf{P}^d_k]$. Theoretically, this probability is calculated by solving the following integral.

$$\Pr[g_j(\mathbf{X},\mathbf{P}) \le 0 | \mathbf{X}^d_k,\mathbf{P}^d_k] = F^c_{g_j,k}(0) = \int_{g_j \le 0 | [\mathbf{X}^d_k,\mathbf{P}^d_k]} f^c_{XP} d\mathbf{X}^c d\mathbf{P}^c \tag{4}$$

The reliability of g_j is then as shown in Eq.(5).

$$\Pr[g_j(\mathbf{X},\mathbf{P}) \le 0] = F_{g_j}(0) = \sum_{k=1}^{D} f^d_{XP,k} F^c_{g_j,k}(0) \tag{5}$$

Figure 2 shows the graphical representation of Eq.(5) on the g_j axis with probability density as the ordinate (unlike the discrete stem plot). Reliability of g_j is then the sum of the areas under the curves to the left of and including the constraint boundary. It should be emphasized that these curves are not the densities curves of the conditional probability in Eq.(4). Rather, they are the *weighted* conditional density curves of the probability in Eq.(5) in that each curve represents one multiplication element of the summation. As such, the area under the k-th curve is not 1, but $f^d_{XP,k}$, $k = 1,\ldots,D$.

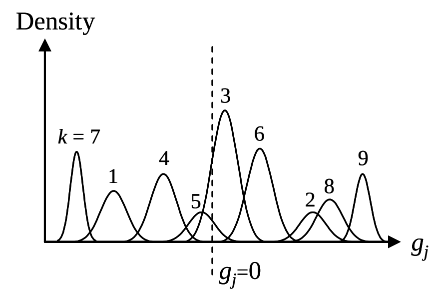

Figure 2. Graphical representation of mix continuous-discrete reliability

Analytical solution to the integral in Eq.(4) is generally not possible while numerical integration is prohibitively inefficient. In the next subsections we discuss three alternative methods to calculate mixed continuous-discrete reliability: (1) Monte Carlo Analysis (MCA), (2) Full Factorial Reliability Analysis (FFRA), and (3) Partial Factorial Reliability Analysis (PFRA).

3.1. MONTE CARLO ANALYSIS

The simplest alternative to numerical integration is to use a Monte Carlo simulation to approximate it. In this method, the continuous components of the problem are first discretized followed by a large number of random experiments. The reliability is then approximately equal to the ratio of feasible experiments to the total number of experiments.

There are two approaches to use Monte Carlo simulation to calculate mixed continuous-discrete reliability. One, we use it to approximate $F^c_{g_j,k}(0)$ in Eq.(4) only, and then repeat it D times for each discrete factorial. We can then use the approximate $F^c_{g_j,k}(0)$ in Eq.(5) to calculate reliability. Two, we use Monte Carlo simulation to directly approximate the continuous-discrete reliability in Eq.(5). The dimension of the first and second approach is $(n_c + m_c)$ and $(n_c + m_c) + (n_d + m_d)$, respectively. So, in general the second approach requires more experiments for the same accuracy. However, the first approach needs to be repeated D times. Since D is much larger than $(n_d + m_d)$, the second approach is generally more efficient than the first. For the rest of the article we will use the term Monte Carlo Analysis to refer to the second approach.

MCA provides an excellent approximation to Eq.(5) provided the number of random experiments is large enough. In the absence of analytical solution, MCA results are often regarded as the “exact” solutions. However, the large number of function evaluations needed by the method prevents its widespread use in practical applications.

3.2. FULL FACTORIAL RELIABILITY ANALYSIS

An alternative method to calculate mixed continuous-discrete reliability is to use the First Order Reliability Method (FORM) [8–10] to approximate the integral in Eq.(4). In this method we calculate $F^{\mathrm{c}}_{g_j,k}(0)$ using FORM at all discrete factorials, and then use them in Eq.(5).

FORM has been reported to be efficient in terms of function evaluations [8, 9]. However, here we have to perform FORM D times, once for each discrete factorial. Since D is generally large, the total number of function evaluations needed is still very high. If we can somehow limit the use of FORM only to those discrete factorials that really need it, we can drastically reduce the number of function evaluations required. A method to filter the discrete factorials will be discussed next.

3.3. PARTIAL FACTORIAL RELIABILITY ANALYSIS

In FFRA, the integral in Eq.(4) is calculated at all discrete factorials. Mathematically, however, we really only need to calculate $F^{\mathrm{c}}_{g_j,k}(0)$'s at those discrete factorials that are close to the constraint boundary and whose $f^{\mathrm{d}}_{XP,k}$'s are large. If $g_{j,k}$ at a k-th discrete factorial (with the means $[\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}}]$ for the continuous components) is far from the boundary, then $F^{\mathrm{c}}_{g_j,k}(0)$ can be approximated with $I_{j,k}$ with a very small error. Similarly, if $f^{\mathrm{d}}_{XP,k}$ is very low, $F^{\mathrm{c}}_{g_j,k}(0)$ can also be approximated with $I_{j,k}$ with a maximum error of only $(0.5) f^{\mathrm{d}}_{XP,k}$.

For illustration, consider the following hypothetical discrete density distributions with $D = 4$ as shown in Fig 3. For the $k = 1$ distribution, $f^{\mathrm{d}}_{XP,1}$ is very small and $g_{j,1}$ is far from the boundary. So for this distribution, $F^{\mathrm{c}}_{g_j,1}(0) \approx I_{j,1} = 1.0$ with very little error. Likewise, $f^{\mathrm{d}}_{XP,3}$ is also very small and its contribution to the overall reliability is small. So although it is close to the boundary, the error induced by approximating $F^{\mathrm{c}}_{g_j,3}(0) \approx I_{j,3} = 0$ is small. In contrast, the $k = 2$ distribution is very close to the boundary and its $f^{\mathrm{d}}_{XP,2}$ is large. For this distribution $I_{j,2} = 1.0$ is not a good approximation to $F^{\mathrm{c}}_{g_j,2}(0)$, and we need to use FORM to calculate it more accurately. Similarly for the $k = 4$ distribution, although it is far from the boundary, $f^{\mathrm{d}}_{XP,4}$ is so large that we cannot neglect the lower tail portion of the distribution, and we need to calculate more accurately.

From the previous discussion, it is apparent that we need two pieces of information to determine if it is necessary to use FORM at the k-th factorial: (1) $f^{\mathrm{d}}_{XP,k}$ value, and (2) relative distance of $g_{j,k}$ from the constraint boundary. The first piece of information is already available as part of our problem formulation and assumption; the second piece of information is not as readily available.

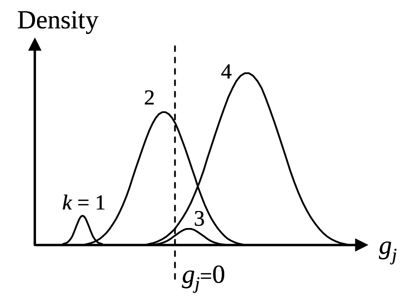

Figure 3. Different importance of density distributions

To determine how far $g_{j,k}$ is from the constraint boundary, we define an *influence function* $h_{j,k}(g_{j,k}) : \mathbb{R}^1 \to \mathbb{R}^1$ to be a monotonically decreasing mapping of the distance from the constraint boundary. The function is maximum when $g_{j,k} = 0$, and gradually decreases as $g_{j,k}$ moves away from the boundary. Many types of influence function can be used, e.g., Gaussian or parabolic. In this PFRA method we use the following influence function:

$$h_{j,k}(g_{j,k}) = \begin{cases} 1 - \Phi(-g_{j,k}/\hat{\sigma}_{g_j}) & \text{if} \quad g_{j,k} \le 0 \\ \Phi(-g_{j,k}/\hat{\sigma}_{g_j}) & \text{if} \quad g_{j,k} > 0 \end{cases} \tag{6}$$

Where $\Phi(.)$ is the cdf of a standard normal distribution with zero mean and standard deviation of 1, and $\hat{\sigma}_{g_j}$ is a user-specified parameter. A nice feature of this influence function is that it readily provides an approximation to $F^{\mathrm{c}}_{g_j,k}(0)$. If we assume the weighted conditional densities are all normal distributions with a standard deviation of $\hat{\sigma}_{g_j}$, then:

$$F^{\mathrm{c}}_{g_j,k}(0) \approx \hat{F}^{\mathrm{c}}_{g_j,k} = \begin{cases} 1 - h_{j,k} & \text{if} \quad g_{j,k} \le 0 \\ h_{j,k} & \text{if} \quad g_{j,k} > 0 \end{cases} \tag{7}$$

The $h_{j,k}$ function in Eq.(6) has a maximum value of 0.5 at $g_{j,k} = 0$, and gradually decays with a distance from the boundary according to the error function $erf(.)$. The decay rate of the function depends on the specified $\hat{\sigma}_{g_j}$, the larger it is the slower the decay. Figure 4 shows the graphs of $h_{j,k}$ for three values of $\hat{\sigma}_{g_j}$.

A large $\hat{\sigma}_{g_j}$ places more importance to density curves far away from the boundary and increases the number of discrete factorials chosen for FORM analysis. This in turn will increase the total number of function evaluations. In general, however, the accuracy of the calculated reliability will also improve. In contrast, a small $\hat{\sigma}_{g_j}$ will decrease the total number of function evaluations, but generally at the expense of accuracy. In our implementation, we use the value $\hat{\sigma}_{g_j} = 0.1$. One very important remark regarding this choice of value: since $\hat{\sigma}_{g_j}$ is our approximation to the density curves' standard deviations, it is critical that g_j is numerically scaled to be within the same order of magnitude. In engineering problems, a typical scaling procedure is to use the upper/lower bound of the constraints as the normalizing factor.

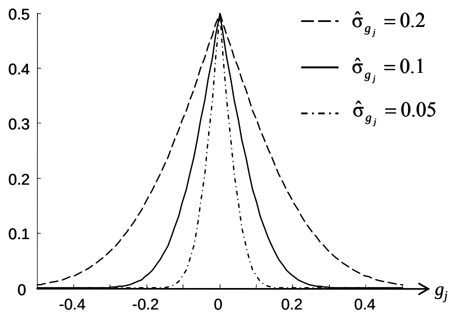

Figure 4. Influence function for different $\hat{\sigma}_{g_j}$

Using $f^{\mathrm{d}}_{XP,k}$ and $h_{j,k}$, we can now determine if the k-th discrete factorial is important enough to warrant a FORM analysis. We define a quantity $H_{j,k}(g_{j,k}) = f^{\mathrm{d}}_{XP,k} h_{j,k}$ to be the *importance function* of the k-th factorial. If $H_{j,k}$ is large, then it is necessary to perform FORM at this particular factorial. If $H_{j,k}$ is small, then we can use $\hat{F}^{\mathrm{c}}_{g_j,k}$ as an approximation.

The step-by-step algorithm of the PFRA method to calculate mixed continuous-discrete reliability of the j-th constraint is as follows:

Step 1. Set $\hat{\sigma}_{g_j} = 0.1$ and $S\bar{H}_{\min} = 0.95$.
Step 2. Calculate $g_{j,k}$, $h_{j,k}$, and $H_{j,k}$ for all $k = 1,\ldots,D$. Calculate the approximation $\hat{F}^{\mathrm{c}}_{g_j,k}$.
Step 3. Calculate the sum of all importance values $SH = \sum_{k=1}^{D} H_{j,k}$.
Step 4. If $SH \leq 0.001$, approximate the reliability $F_{g_j}(0) \approx \sum_{k=1}^{D} f^{\mathrm{d}}_{XP,k}\hat{F}^{\mathrm{c}}_{g_j,k}$, then stop. Else continue.
Step 5. Calculate the normalized importance value $\bar{H}_{j,k} = H_{j,k}/SH$ for all $k = 1,\ldots,D$.
Step 6. Sort the $\bar{H}_{j,k}$'s from largest to smallest. Start from the largest $\bar{H}_{j,k}$, select $NL \leq D$ discrete factorials such that $\sum_{i=1}^{NL} \bar{H}_{j,i} \geq S\bar{H}_{min}$.
Step 7. From the selected NL factorials, discard the $ND \leq NL$ factorials whose $\bar{H}_{j,k} \leq 0.001$.
Step 8. For the selected $NP = NL - ND$ factorials, use FORM to calculate $\hat{F}^{\mathrm{c}}_{g_j,k}$. For the rest of the factorials, keep the $\hat{F}^{\mathrm{c}}_{g_j,k}$ computed in Step 2.
Step 9. The approximate reliability of the constraint is $F_{g_j}(0) \approx \sum_{k=1}^{D} f^{\mathrm{d}}_{XP,k}\hat{F}^{\mathrm{c}}_{g_j,k}$. Stop.

There are two user-specified parameters in the algorithm: $\hat{\sigma}_{g_j}$ and $S\bar{H}_{\min}$. The effect of $\hat{\sigma}_{g_j}$ on the total number of function evaluations and the accuracy of the result has been discussed previously. The parameter $S\bar{H}_{\min}$ controls the minimum percentage of total $H_{j,k}$'s that are selected. The larger $S\bar{H}_{\min}$ the more discrete factorials will be selected. This will increase the accuracy of the approximation, but with an increase in the number of function evaluations. A decrease in $S\bar{H}_{\min}$ will result in an opposite effect. In our implementation, we set $S\bar{H}_{\min} = 0.95$, i.e., 95% of the total $H_{j,k}$'s are selected.

The algorithm above contains two filtering steps: *negligible importance* filtering (Step 4), and *concentrated importance* filtering (Step 7). Step 4 in the algorithm accounts for the case where all the density curves are far from the constraint boundary, as indicated by the low SH value (see Fig.5). In this case, the $\hat{F}^{\mathrm{c}}_{g_j,k}$ approximation is sufficient for all discrete factorials, and no FORM analysis is needed at all. Step 7 in the algorithm accounts for the case where only a few of the NL discrete factorials selected in Step 6 are really important (see Fig.6). This is indicated by the very small $\bar{H}_{j,k}$'s of the other ND factorials.

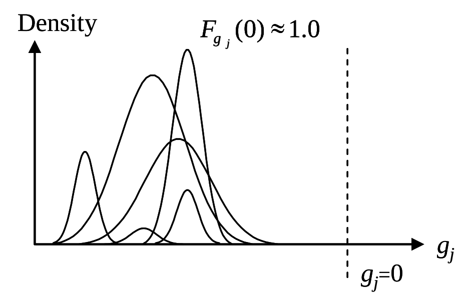

Figure 5. Illustration of negligible importance

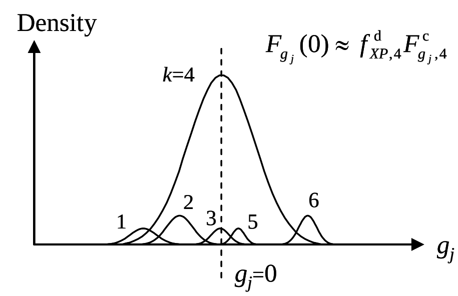

Figure 6. Illustration of concentrated importance

Note that in the above algorithm we have used a normal distribution to approximate $F^{\mathrm{c}}_{g_j,k}(0)$. If the actual weighted conditional densities are non-symmetric, then other forms of distributions might give better approximations.

3.4. ERROR AND EFFICIENCY

Of the three methods to calculate reliability presented, MCA is the most accurate followed by FFRA and PFRA. As mentioned before, when analytical solution is not available, MCA solution is often regarded as the true solution. If the curvature of the constraint function is not too large, FFRA solution will also be very close to that of MCA's. Unfortunately, the FFRA error due to FORM linearization is problem dependent, and there is no analytical estimation to it. Nevertheless, the error induced by FORM has been reported to be small [8, 9].

Since PFRA uses FORM, it is also affected by the linearization error. In addition to this error, PFRA also induces an error by approximating the conditional probabilities at some of the discrete factorials. Like the FORM error, the actual amount of this error is problem dependent and is not possible to formulate analytically. In the best case, all the density curves are far from the boundary (negligible importance case), and the PFRA error is zero. In the worst case, however, the normal approximation and $\hat{\sigma}_{g_j}$ estimate might be very different from the actual density curves. In this situation, the maximum error of the conditional probabilities not calculated by FORM is $(0.5)\,f^{\mathrm{d}}_{XP,k}$. Since we select NP factorials for FORM analysis, the total maximum error is:

$$e_{\max} = \sum_{k=1}^{D-NP} (0.5)\,f^{\mathrm{d}}_{XP,k} \tag{8}$$

This error does not include the FORM linearization error. Also, Eq.(8) involves the quantity NP that is problem dependent, and so the error calculation is also problem dependent. Since we impose the $\overline{SH}_{\min} = 0.95$ selection criterion in the PFRA algorithm (Step 6), NP is generally large and the $f^{\mathrm{d}}_{XP,k}$'s at the $(D-NP)$ factorials are small. So overall the maximum error is small.

Although MCA is the most accurate, it is also the most inefficient of the three methods in terms of number of function evaluations (FE). The actual magnitude of MCA's FE depends on the MCDRO's dimension, but $FE_{\mathrm{MCA}} > 10^6$ is typical. For the FFRA method, $FE_{\mathrm{FFRA}} = (D)FE_{\mathrm{FORM}}$ where FE_{FORM} is the number of function evaluations performed by FORM. There is no analytical formula for FE_{FORM}, but it is reported to be in the order of $O(10^0)$ to $O(10^2)$ [13]. Due to the very efficient FORM, in general FFRA is much more efficient than MCA, even though FE_{FFRA} involves the factor D.

PFRA uses FORM at some, but not all, of the D discrete factorials. The FE of PFRA is $FE_{\mathrm{PFRA}} = D + \alpha(D)FE_{\mathrm{FORM}}$, where $0 \le \alpha \le 1$. The D in the first factor of the sum accounts for the evaluations of $g_{j,k}$ at all $k = 1,\ldots,D$ (Step 2). In the best case (negligible importance case), $\alpha = 0$ and $FE_{\mathrm{PFRA}} = D$, i.e., no FORM analysis is needed at all. In the worst case, PFRA performs FORM at all discrete factorials. For this case, $\alpha = 1$ and $FE_{\mathrm{PFRA}} = D(1+FE_{\mathrm{FORM}}) \approx FE_{\mathrm{FFRA}}$ for $FE_{\mathrm{FORM}} \gg 1$, i.e., PFRA is as efficient as FFRA. The quantity α decreases as the design point moves away from the constraint boundary, so PFRA is more efficient for design points far from the boundary (as is usually the case in the early iterations of an optimization run). Overall, PFRA is at least as efficient as FFRA, while MCA is the least efficient of all.

A study of the error and efficiency properties of MCA, FFRA, and especially PFRA is given in the next section through examples.

4. DEMONSTRATION EXAMPLES

We apply the PFRA algorithm to two examples. In the first, we show a step-by-step implementation of the algorithm to calculate a mixed continuous-discrete reliability of a design point with respect to a single inequality constraint. In the second, we show the application of PFRA in an optimization algorithm to solve a MCDRO problem.

4.1. SINGLE CONSTRAINT RELIABILITY

Consider a quadratic constraint g_1 that is a function of one continuous variable X_1, one integer variable X_2, and one integer parameter P. The variables and parameters in this problem are random. Following the notation in Section 2, $\mathbf{X}^{\mathrm{c}} = [X_1]$ and $\mathbf{X}^{\mathrm{d}} = [X_2]$, where $n_c = 1$ and $n_d = 1$; for the parameters, $\mathbf{P}^{\mathrm{c}} = [\ ]$ and $\mathbf{P}^{\mathrm{d}} = [P]$, where $m_c = 0$ and $m_d = 1$; and so $\mathbf{Y}^{\mathrm{c}} = [X_1]$ and $\mathbf{Y}^{\mathrm{d}} = [X_2, P]$. The objective of this study is to calculate the reliability of a design point whose means are $\boldsymbol{\mu}_{\mathbf{X}} = [\mu_{X_1}, \mu_{X_2}] = [5.5, 7]$ with respect to g_1:

$$\begin{aligned} g_1(\mathbf{X},\mathbf{P}) = {} & \tfrac{1}{350}(7X_1^2 + 6X_2^2 + 8P^2 - 6X_1P + 4X_2P \\ & -15.8X_1 - 93.2X_2 - 63P) + 1 \le 0 \end{aligned} \tag{9}$$

Randomness in the variables and the parameter is as follows (all are independent). The variable X_1 is randomly distributed according to the normal pdf $f_{X_1} = N(\mu_{X_1}, 0.2)$. The integer variable X_2 is distributed around its nominal value within the set $W_1 = \{5,6,7,8,9\}$ according to pmf $f_{X_2} = \{0.05, 0.15, 0.6, 0.15, 0.05\}$. The integer parameter P is distributed within the set $S_1 = \{2,3,4,5\}$ with pmf $f_P = \{0.05, 0.3, 0.45, 0.2\}$. Following our notation, $B_1 = 5$, $C_1 = 4$, and $D = B_1C_1 = 20$. The joint pmf of X_2 and P is shown below (f_{X_2} as columns, f_P as rows).

$$f^{\mathrm{d}}_{XP} = \begin{bmatrix} 0.0025 & 0.0075 & 0.03 & 0.0075 & 0.0025 \\ 0.015 & 0.045 & 0.18 & 0.045 & 0.015 \\ 0.0225 & 0.0675 & 0.27 & 0.0675 & 0.0225 \\ 0.01 & 0.03 & 0.12 & 0.03 & 0.01 \end{bmatrix} \tag{10}$$

The step-by-step implementation of the PFRA algorithm to calculate reliability is as follows:

Step 1. Set $\hat{\sigma}_{g_1} = 0.1$ and $\overline{SH}_{\min} = 0.95$.

Step 2. The g_1 values at all factorials are calculated by substituting $\mu_{X_1} = 5.5$ and the $[X_2, P]$ permutation to Eq.(9). Using g_1, the h_1 values are calculated using Eq.(6). The importance values H_1 are obtained by multiplying h_1 with the joint pmf f^{d}_{XP} in Eq.(10). The approximation $\hat{F}^{\mathrm{c}}_{g_1}$ at each factorial is calculated using Eq.(7). The calculated H_1 and $\hat{F}^{\mathrm{c}}_{g_1}$ are shown in Tables 1 and 2, respectively.

Step 3. The sum of the $H_{1,k}$'s in Table 1 is $SH = \sum_{k=1}^{20} H_{1,k} = 0.2878$.

Step 4. From Step 3, $SH > 0.001$. So this is not a negligible case; continue to Step 5.

Table 1. H_1 values for $[X_2, P]$ factorials

	$X_2 = 5$	6	7	8	9
$P = 2$	0.0003	0.0022	0.0108	0.0023	0.0004
3	0.007	0.0163	0.0591	0.0191	0.0052
4	0.007	0.0141	0.0582	0.0227	0.0087
5	0.0027	0.0063	0.0301	0.0127	0.0026

Table 2. $\hat{F}^{c}_{g_1}$ values for $[X_2, P]$ factorials

	$X_2 = 5$	6	7	8	9
$P = 2$	0.134	0.287	0.361	0.311	0.165
3	0.468	0.638	0.672	0.577	0.344
4	0.688	0.791	0.784	0.663	0.387
5	0.727	0.791	0.75	0.577	0.264

Table 3. $\bar{H}_1$ values for $[X_2, P]$ factorials

	$X_2 = 5$	6	7	8	9
$P = 2$	0.001	0.0076	**0.0375**	0.008	0.0014
3	**0.0243**	**0.0566**	**0.2054**	**0.0664**	**0.0181**
4	**0.0243**	**0.049**	**0.2022**	**0.0789**	**0.0302**
5	0.0094	**0.0219**	**0.1046**	**0.0441**	0.009

Step 5. The normalized importance values $\bar{H}_1$ is shown in Table 3.

Step 6. Starting from the largest $\bar{H}_1$ in Table 3, $NL = 14$ factorials are selected such that $\sum_{i=1}^{14} \bar{H}_{1,k} \geq S\bar{H}_{\min}$. The factorials selected are highlighted in Table 3.

Step 7. From the 14 factorials selected, none has a $\bar{H}_{1,k} \leq 0.001$. So $ND = 0$.

Step 8. For the final $NP = NL - ND = 14$ factorials selected, we use FORM to calculate $\hat{F}^{c}_{g_1}$. For the other factorials, we keep the $\hat{F}^{c}_{g_1}$ values from Table 2. The revised $\hat{F}^{c}_{g_1}$ values are shown in Table 4.

Step 9. The approximate reliability is $F_{g_1}(0) \approx \sum_{k=1}^{20} f^{d}_{XP,k} \hat{F}^{c}_{g_1,k} = 0.8743$. Stop.

As a benchmark, we calculate the reliability using the MCA method in which 1,000,000 samples are taken from the $[X_2, P]$ discrete sets and the discretized X_1 distribution. The MCA reliability value is $F_{g_1}(0)|_{\text{MCA}} = 0.8701$. This value is considered to be the "actual" value. The absolute error of PFRA result is $e = 0.0042$, less than the maximum error predicted in Eq.(8): $e_{\max} = 0.0313$. The relative error of PFRA is:

$$\varepsilon = \frac{F_{g_1}(0)|_{\text{PFRA}} - F_{g_1}(0)|_{\text{MCA}}}{F_{g_1}(0)|_{\text{MCA}}} = 0.0048$$

less than 1%. For comparison, we also calculate the reliability using the FFRA method. The computed $F_{g_1}(0)|_{\text{FFRA}} = 0.8695$ is very close to $F_{g_1}(0)|_{\text{MCA}}$. The discrepancy between the two can be attributed to FORM linearization error as well as round-off error.

Table 4. Revised $\hat{F}^{c}_{g_1}$ values for $[X_2, P]$ factorials

	$X_2 = 5$	6	7	8	9
$P = 2$	0.134	0.287	0.094	0.311	0.165
3	0.369	0.915	0.956	0.777	0.043
4	0.984	0.999	0.999	0.968	0.077
5	0.727	0.999	0.999	0.849	0.264

Table 5. Comparison of MCA, FFRA, and PFRA results

(μ_{X_1}, μ_{X_2})	MCA	FFRA	PFRA	ε
(1.7,6)	1	1	0.9974	-0.0026
(5.5,7)	0.8701	0.8695	0.8743	0.0048
(4.8,9)	0.6704	0.67	0.6679	-0.0037
(2.5,3)	0.5564	0.5563	0.558	0.0029
(1.9,3)	0.4039	0.4046	0.4042	0.0007
(0.9,10)	0.339	0.339	0.3373	-0.005
(2.3,2)	0.1268	0.1274	0.1267	-0.0008
(4.1,2)	0.0788	0.0787	0.0782	-0.0076

Table 5 shows a comparison of the $F_{g_1}(0)$ values and the relative errors obtained using MCA, FFRA, and PFRA for different design points. Table 6 shows the number of function evaluations performed by each method. The percent reduction in function evaluations from using PFRA compared to FFRA:

$$\delta_{\text{red}} = \frac{FE_{\text{FFRA}} - FE_{\text{PFRA}}}{FE_{\text{FFRA}}}$$

is also shown.

As seen in Table 5, PFRA provides an excellent estimate to MCA results for all ranges of reliability values. The relative error of the approximation is less than 1% for all 8 design points. At the same time, using PFRA results in a 14% to 44% reduction in

Table 6. Comparison of FE_{MCA}, FE_{FFRA}, and FE_{PFRA}

(μ_{X_1}, μ_{X_2})	FE_{MCA}	FE_{FFRA}	FE_{PFRA}	δ_{red}
(1.7,6)	1,000,000	129	111	0.14
(5.5,7)	1,000,000	373	250	0.33
(4.8,9)	1,000,000	279	173	0.38
(2.5,3)	1,000,000	141	79	0.44
(1.9,3)	1,000,000	151	115	0.238
(0.9,10)	1,000,000	228	164	0.281
(2.3,2)	1,000,000	112	76	0.321
(4.1,2)	1,000,000	136	82	0.397

the number of function evaluations compared to using FFRA as shown in Table 6.

4.2. DESIGN OF A BELLEVILLE SPRING

The objective of this problem is to optimize the Belleville spring shown in Figure 7 for maximum rated load P_{load}. This example is originally formulated by Siddall [28], and is modified here to be a MCDRO problem. All random variables and parameters are assumed independent.

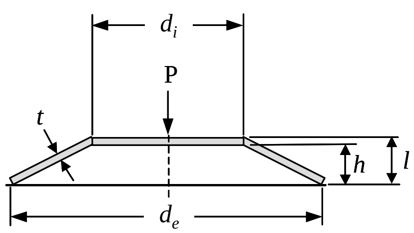

Figure 7. A Belleville spring

The design variables are: external diameter (d_{e}), internal diameter (d_{i}), free height (h), and thickness (t). The variables $[d_{\text{e}}, d_{\text{i}}, h]$ are continuous (in meters), but due to manufacturing practices the sheet metal thickness (t) is only available in multiples of 0.25 mm. The continuous variables $[d_{\text{e}}, d_{\text{i}}, h]$ are random, and each is modeled as a normal distribution with a standard deviation of 0.0866 mm, 0.0767 mm, and 0.0333 mm, respectively [29]. The spring thickness is also random and is discretely distributed according to the following probability.

$$\Pr[t=\tau] = \begin{cases} 0.5, & \tau = \mu_t \\ 0.2, & \tau = \mu_t \pm 0.25 \\ 0.05, & \tau = \mu_t \pm 0.5 \end{cases} \tag{11}$$

The spring is to be made from high strength steel, but it is uncertain which particular grade of steel will be ultimately used. Table 7 shows the four choices of steels for the spring, their properties, and the probabilities of each choice. In this table, the quantities E and σ_{aw} are the elastic modulus and allowable stress of the steels, respectively.

Table 7. Steel properties and their probabilities

Steel type	E (GPa)	σ_{aw} (MPa)	Pr
A	190	1100	0.1
B	200	1350	0.25
C	205	1500	0.6
D	210	1600	0.05

So in this problem $\mathbf{X}^{\text{c}} = [d_{\text{e}}, d_{\text{i}}, h]$ and $\mathbf{X}^{\text{d}} = [t]$, and $n_c = 3$ and $n_d = 1$. For the parameters, $\mathbf{P}^{\text{c}} = [\ ]$ and $\mathbf{P}^{\text{d}} = [\text{steel type}]$, and $m_c = 0$ and $m_d = 1$. The continuous components are $\mathbf{Y}^{\text{c}} = [d_{\text{e}}, d_{\text{i}}, h]$, while the discrete components are $\mathbf{Y}^{\text{d}} = [t, \text{steel type}]$. The pdf's of the continuous variable are $f_{d_{\text{e}}} = N(\mu_{d_{\text{e}}}, 0.0866)$, $f_{d_{\text{i}}} = N(\mu_{d_{\text{i}}}, 0.0767)$, and $f_h = N(\mu_h, 0.0333)$. For the discrete variable, $W_1 = \{\mu_t - 0.5, \mu_t - 0.25, \mu_t, \mu_t + 0.25, \mu_t + 0.5\}$ with a pmf $f_t = \{0.05, 0.2, 0.5, 0.2, 0.05\}$ and $B_1 = 5$. For the discrete parameter: $S_1 = \{\text{A,B,C,D}\}$ with a pmf $f_{\text{steel}} = \{0.1, 0.25, 0.6, 0.05\}$ and $C_1 = 4$. The full factorial of the discrete components is $D = B_1 C_1 = 20$.

The spring optimization is constrained by two design constraints, maximum allowable stress and maximum mass, and five geometric constraints. The formulation of the deterministic MCDO is shown in Eq.(12) (notice how the constraints are numerically scaled). Here P_{load} is the rated load (N), $\sigma_{\max}$ is the maximum stress (Pa), and m is the spring mass (kg). Details of the deterministic problem can be found in [30].

$$\begin{aligned} \underset{\mathbf{x}=[d_{\text{e}}, d_{\text{i}}, h, t]}{\text{minimize}} \quad & f(\mathbf{x}, \mathbf{p}) = P_{\text{load}} \\ \text{subject to:} \quad & g_1(\mathbf{x}, \mathbf{p}) \equiv \sigma_{\max}/\sigma_{\text{aw}} - 1 \leq 0 \\ & g_2(\mathbf{x}, \mathbf{p}) \equiv m/m_{\max} - 1 \leq 0 \\ & g_3(\mathbf{x}, \mathbf{p}) \equiv h_{\min}/h - 1 \leq 0 \\ & g_4(\mathbf{x}, \mathbf{p}) \equiv (h+t)/l - 1 \leq 0 \\ & g_5(\mathbf{x}, \mathbf{p}) \equiv d_{\text{e}}/d_{\max} - 1 \leq 0 \\ & g_6(\mathbf{x}, \mathbf{p}) \equiv 1.25 d_{\text{i}}/d_{\text{e}} - 1 \leq 0 \\ & g_7(\mathbf{x}, \mathbf{p}) \equiv 1 - 0.3(d_{\text{e}} - d_{\text{i}})/h \leq 0 \end{aligned} \tag{12}$$

The optimum of the deterministic MCDO is $[d_{\text{e}}, d_{\text{i}}, h, t]^* = [0.3, 0.232, 5.0, 8.0]$ (using steel type C as the material). The t^* and h^* values shown are in mm. The maximum load of this deterministic optimum is $P^*_{\text{load}} = 67.82$ kN. The constraint values are $\mathbf{g}^* = [-0.0019, -0.117, 0, -0.35, 0, -0.0307, -3.038]$ where g_3 and g_5 are active. Based on the specified pdf's and pmf's, the reliabilities of this optimal point w.r.t. the constraints are $\boldsymbol{F}^*_{g_j}(0) = [0.376, 0.994, 0.5, 1.0, 0.976, 0.982, 1.0]$. As can be

Table 8. Comparison of MCA, FFRA, and PFRA optima

	MCA	FFRA	PFRA
d_e (m)	0.297	0.289	0.276
d_i (m)	0.22	0.2065	0.168
h (mm)	5.07	5.08	5.44
t (mm)	6	6	6
P_{load} (kN)	26.068	25.375	23.587
$F_{g_1}(0)$	0.995	0.996	0.989
$F_{g_3}(0)$	0.983	0.992	1.0
FE	4,470,000	177,752	110,866

seen, the deterministic optimum has low reliabilities in terms of g_1 and g_3.

4.2.1. RELIABILITY OPTIMIZATION

For reliability optimization, all constraints are replaced with probabilistic ones. The lower reliability bound for all probabilistic constraints is set to be $R_j = 0.99$ for $j = 1, \ldots, 7$.

We solved the MCDRO using the MOST algorithm as implemented in the commercial software iSIGHT 8.0. MOST is an optimization algorithm that combines SQP and branch-and-bound algorithms to solve mixed continuous-discrete problems [31]. For comparison, we solved the problem three times, each using MCA, FFRA, and PFRA for reliability calculation. For fairness, the same starting point $[d_e, d_i, h, t]^* = [0.3, 0.2, 5.0, 7.0]$ is used in all three runs. For the MCA method, 1,000 samples are used (this number provides a relatively accurate reliability while still practically manageable). For the FFRA method, the HL-RF algorithm [9] is chosen for FORM calculation. For the PFRA method, the parameters are specified to be $\hat{\sigma}_{g_j} = 0.1$ and $S\bar{H}_{\min} = 0.95$. The $\hat{F}^c_{g_j}$ values at the selected factorials are also calculated with FORM using the HL-RF algorithm.

Table 8 shows the reliability optima obtained and the total number of function evaluations performed. In counting FE, each calculation of either the objective or constraint function is considered one evaluation. The table also shows the reliabilities of the optima in terms of g_1 and g_3. Reliability values in terms of the other constraints are the same for all three optima.

We see in Table 8 that MCA optimum has the highest P_{load} while PFRA optimum has the lowest. However, we also see that MCA optimum does not quite satisfy the third reliability constraint. A possible explanation is that the number of samples used is not large enough. Even with the relatively low number of samples, however, the optimization already required more than 4 million function evaluations. This observation further demonstrates the impracticality of MCA for practical problems. FFRA's P_{load} is slightly lower than that of MCA's, but it satisfies all the reliability constraints. In terms of efficiency, FE_{FFRA} value is also significantly lower than FE_{MCA}. PFRA's P_{load} is lower than FFRA's. This may be caused by the inconsistent changes in the reliability values calculated using PFRA. Nevertheless, in return for this 7% decrease in P_{load}, we gain a significantly larger 37% decrease in total function evaluations. These results suggest a potential hybrid optimization algorithm utilizing both PFRA and FFRA to achieve convergence and efficiency.

5. SUMMARY

This article presents a method to reduce the number of function evaluations needed to calculate a mixed continuous-discrete reliability while maintaining accuracy. Unlike the FFRA method, PFRA uses FORM to calculate the continuous cumulative probabilities only at some of the discrete factorials. These discrete factorials are systematically selected based on their importance, which in turn depends on the relative distance from the boundary and the probability of the discrete components. In the numerical example, the PFRA result is found to be in excellent agreement with the MCA value along with a significant improvement in computational efficiency. The relative error of the approximation is less than 1%, and the number of function evaluations decreases by as much as 44%. When used in reliability optimization, PFRA is also found to perform well. The PFRA optimum is slightly inferior to the FFRA optimum (7% lower), but the total number of function evaluations is decreased by 37%.

ACKNOWLEDGMENT

This research was partially supported by the Automotive Research Center (ARC), a U.S. Army Center of Excellence in Modeling and Simulation of Ground Vehicles at the University of Michigan. This support is gratefully acknowledged. The opinions expressed in the article are those of the authors and do not necessarily reflect those of the sponsors.

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