AIAA JOURNAL
Vol. 41, No. 5, May 2003

# Convergence Properties of Analytical Target Cascading

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**Analytical target cascading is a relatively new methodology for the design of engineering systems. Analytical target cascading deals with the issue of propagating desirable top-level product design specifications (or targets) to appropriate targets at lower levels in a consistent and efficient manner. Most existing problem formulations for multilevel design often exhibit convergence difficulties. It is proved that the analytical target cascading process converges to a point that satisfies the necessary optimality conditions of the original design target problem.**

## Nomenclature

$\mathcal{C}_{ij}$ = set of children of the element $j$ in the set $\mathcal{E}_i$
$c_{ij}$ = number of children of the element $j$ in the set $\mathcal{E}_i$ or cardinality of set $\mathcal{C}_{ij}$
$\mathcal{E}_i$ = collection of all elements at the $i$th level
$\boldsymbol{F}$ = forest of the problem between the $p$th and $r$th levels
$f$ = objective function
$\boldsymbol{g}$ = design inequality constraint functions
$H$ = submatrix of identity matrix
$\boldsymbol{h}$ = design equality constraint functions
$I_n$ = $n$-dimensional identity matrix
$J$ = Jacobian matrix of design constraint functions
$n$ = number of design variables in problem corresponding to forest $\boldsymbol{F}$
$p$ = number of constraints
$\boldsymbol{R}$ = system response functions
$\mathbb{R}$ = set of real numbers
$\mathbb{R}^n$ = $n$-dimensional Euclidean space
$\boldsymbol{T}$ = design specifications or targets
$T_a(\boldsymbol{x}^*)$ = set of indices corresponding to the active inequality constraints at $\boldsymbol{x}^*$
$w$ = deviation weighting coefficient
$\boldsymbol{x}$ = design variables in design target problem; all variables in problem corresponding to forest $\boldsymbol{F}$
$\boldsymbol{y}_{ij}$ = linking design variables for element $j$ at the $i$th level
$\epsilon$ = tolerance variable for compatibility between two levels of problem hierarchy
$\boldsymbol{\lambda}$ = Lagrange multipliers for design constraints
$\nabla$ = gradient

*Subscripts*

$a$ = active inequality constraint
$i, p, q, r$ = $i$th, $p$th, $q$th, or $r$th level in problem hierachy
$j$ = element in $\mathcal{E}_i$
$\boldsymbol{L}$ = lower subforest of the problem between the $(q+1)$th and $r$th levels
$\boldsymbol{U}$ = upper subforest of the problem between the $p$th and $q$th levels

*Superscripts*

$E$ = equality constraints
$I$ = inequality constraints
$i$ = level in problem hierachy
$R$ = responses
$t$ = transpose
$y$ = linking design variables

## I. Introduction

IN a typical product development process, one of the early steps is the verification that the resulting product or system will meet some predefined design specifications or targets $\boldsymbol{T}$. When it is assumed that analytical or computational capabilities exist to compute the responses $\boldsymbol{R}$ of the system for a given design $\boldsymbol{x}$, the design target problem can be formulated as the mathematical optimization problem

$$\min_x \|\boldsymbol{R}(\boldsymbol{x}) - \boldsymbol{T}\| \tag{1}$$

subject to $\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$ and $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$, where $\boldsymbol{g}$ and $\boldsymbol{h}$ are design constraint functions.

For a complex product, such as an automobile or aircraft, direct solution of problem (1) is not possible. Instead, the overall product targets must be translated to proper targets for the various parts that constitute the product, which are complex products themselves and must be designed in a relatively independent manner. Thus, the design target problem becomes one of propagating (or "cascading") targets throughout a hierarchy representing the decomposition of the product into its parts. The difficulty is that complex product parts are never really independent of each other, and so the targets set for them must be consistent with each other. Moreover, there must be an assurance that if the individual part targets are met, then the overall target for the entire product will be also met. Finally, recognizing that this entire process is likely to be quite complicated, efficient allocation of targets at an early stage is highly desirable.

In the sequel, the main assumption made is that the performance of the product can be analyzed and adequately described by the functions $\boldsymbol{R}$, $\boldsymbol{g}$, and $\boldsymbol{h}$, and that these functions can be computed for any given design $\boldsymbol{x}$. Hence, the term analytical is used to characterize the target cascading process.

Analytical target cascading (ATC)[1–3] is a formal methodology for multidisciplinary optimal design. First, the design target problem is

Received 22 December 2001; revision received 9 December 2002; accepted for publication 9 December 2002. 

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partitioned into a hierarchical set of subproblems associated with the supersystem (that is, the product itself) and the systems, subsystems, and components making up the supersystem. The formulation is general enough to account for any number of levels in this hierarchy. Design specifications (or targets) defined at the top supersystem level are then cascaded down to lower levels following the prescribed ATC process. Once lower-level targets are identified, individual design target subproblems are formulated at each level using more detailed models and complex simulations. Thus, components, subsystems, and systems can be designed to match cascaded targets in a manner consistent with the overall targets.

The main benefits of target cascading are reduction in design-cycle time, avoidance of design iterations late in the development process, and increased likelihood that physical prototypes will be closer to production quality. Target cascading also facilitates concurrency in system design: Once targets are identified for systems, subsystems, and components, the latter elements can be isolated and designed in detail independently, allowing the outsourcing of subsystems and components to suppliers.

ATC has been applied in automotive vehicle design to cascade ride quality and handling specifications utilizing suspension, tire, and spring analysis models.[4] A recent application of ATC[5,6] involves the design of an advanced heavy tactical truck, which has a series hybrid electric powertrain configuration, and emphasizes fuel economy, ride, and mobility characteristics. In an application to vehicle redesign,[7] ATC is used to cascade fuel economy, performance, and ride quality specifications to the suspension, engine, and transmission systems of a U.S. class VI commercial truck. ATC has also been extended to the design of product families with predefined platforms[8] to accommodate the presence of shared systems, subsystems, or components.

Several other formulations for multilevel design of hierarchical systems have been proposed; unfortunately, they often exhibit convergence difficulties. In structural optimization, for example, it is common to take advantage of weakly coupled local structures and to formulate the design problem as a hierarchical problem. Lower-level design subproblems are coupled only through interactions with higher levels.[9–12] One drawback of these formulations is that derivatives of the lower-level optima may be discontinuous functions of the higher-level variables, making the multilevel problem more ill-conditioned than the original problem.[13,14] In collaborative optimization (CO),[15] subsystem analyses are decoupled by introducing compatibility constraints at the system level after reformulating the design problem as a bilevel programming problem. A drawback of this formulation is that the system-level constraint Jacobian either vanishes at all feasible points of the system-level problem or is discontinuous at a solution.[16] Moreover, the convergence behavior of optimization algorithms applied to CO might be erratic.

In this paper, global convergence properties of the ATC formulation, when used together with optimization algorithms, are proven, hence, establishing ATC as a robust formulation for multidisciplinary optimal design. The paper provides an interpretation of the ATC process as a recursive solution process of two "overlapping" problems at a time, which enables the use of the hierarchical overlapping coordination (HOC) strategy.[17,18] This leads to the convergence of ATC when combined with the separable structure of the design constraints, as proven in the lemma in Sec. IV.A. In this regard, what makes ATC convergent is its HOC interpretability with separable constraint structure.

The paper is organized as follows: In Sec. II, a relaxation of the original design target problem is presented, and a description of the ATC process is given. In Sec. III, based on the tree structure of the relaxed design target problem, a detailed description of the design target subproblem for a portion of the problem hierarchy is presented. In Sec. IV, the ATC process is formulated as a coordination between the design target subproblems in the problem hierarchy. In Sec. V, by combination of the interpretation of the ATC process developed in Sec. IV with the hierarchical overlapping coordination strategy,[17,18] the convergence of the ATC process is established.

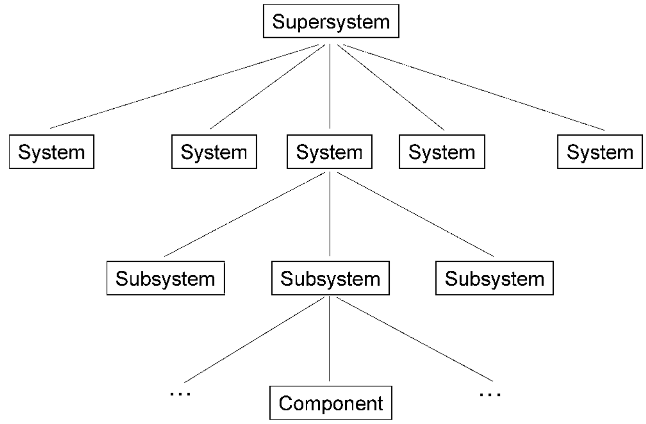


**Fig. 1 Typical decomposition of a (super)system.**

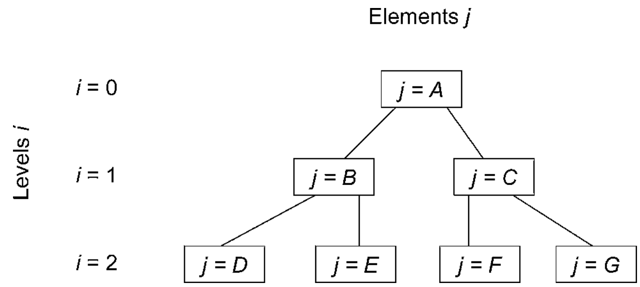


**Fig. 2 Example of index notation for a hierarchically partitioned design problem.**

## II. Analytical Target Cascading and the Design Target Problem

Design of a product, or supersystem, entails determining the values of design variables such that the supersystem meets its design targets. For an automotive vehicle, for example, these targets can be measures of fuel consumption, emissions, performance, handling, ride quality, cost, and so on. ATC assumes that the supersystem and associated models can be hierarchically partitioned into systems, subsystems, and components, with as many levels as needed. Each entity at each level that corresponds to a node of the tree structure is called an element. Figure 1 shows a typical decomposition of a (super) system.

### A. Assumptions and Definitions

The following assumptions are made with respect to the supersystem and the models describing its behavior:

1) Models describing the behavior or response of each element in the problem hierarchy are available at an appropriate level of fidelity. That is, they are models of reduced fidelity at the top, for example, system levels and high fidelity at the bottom, for example, component levels. These models can be analytical or experimental, and/or quantitative or qualitative. Approximate analytical models need to be generated if they are not available.

2) The interaction between two consecutive levels in the hierarchy is similar at all levels except for the top, that is, supersystem and bottom, that is, component, levels. Supersystem/system and subsystem/component interactions are special cases of the more general system/subsystem interaction of intermediate levels. This enables using similar ATC formulations for any two levels of the hierarchy.

3) To represent the hierarchy of the partitioned design problem, the set $\mathcal{E}_i$ is defined as the collection of all elements at the $i$th level. The supersystem level corresponds to the zeroth level, that is, the supersystem corresponds to the element $l \in \mathcal{E}_0$. For each element $j$ in the set $\mathcal{E}_i$, $\mathcal{C}_{ij} := \{k_1, \ldots, k_{c_{ij}}\}$ is defined as the set of its $c_{ij}$ children. An example is presented in Fig. 2: At level $i = 1$ of the hierarchy, we have $\mathcal{E}_1 = \{B, C\}$, and for element $C$ on that level we have $\mathcal{C}_{1C} = \{F, G\}$. Similarly, $\mathcal{E}_2 = \{D, E, F, G\}$.

4) Elements at the same level of the hierarchy having the same parent element can share design variables called linking design variables $\boldsymbol{y}_{ij}$, for element $j$ at the $i$th level. This concept can be extended to elements at the same level of the hierarchy having a common ancestor element.

5) Responses $\boldsymbol{R}_i$ for $i$-level elements can be associated either with (top-level) supersystem targets $\boldsymbol{T}$ or with "cascaded down" or "passed up" targets $\boldsymbol{R}_i^{i-1}$ or $\boldsymbol{R}_i^{i+1}$, respectively. The latter quantities link two successive levels in the design hierarchy according to the ATC process.

6) Responses $\boldsymbol{R}_{ij}$ of a system corresponding to element $j$ at the $i$th level depend on responses $\boldsymbol{R}_{(i+1)k}$, $k \in \mathcal{C}_{ij}$, of the subsystems making up the system, as well as on the system's local design variables $\boldsymbol{x}_{ij}$ and linking design variables $\boldsymbol{y}_{ij}$, that is, $\boldsymbol{R}_{ij} = \boldsymbol{r}_{ij}(\boldsymbol{R}_{(i+1)k_1}, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}, \boldsymbol{x}_{ij}, \boldsymbol{y}_{ij})$. In general, the response or behavior of an element depends on design variables characterizing the element, as well as on responses of (lower-level) elements making up the element.

7) Element constraint functions $\boldsymbol{g}$ and $\boldsymbol{h}$ define the feasible space of the element's design variables and responses.

8) Dummy elements can be introduced in the design hierarchy to account for cases in which element responses depend on responses of elements two or more levels down in the hierarchy.

9) A tree is a connected graph without any circuits, and a forest is a collection of trees.[19] Note that a subgraph of a tree could be a forest. This is the case in Sec. III, where the forests associated with design subproblems are derived from the hierarchy tree of the original problem.

### B. Design Target Problem

In the context of ATC, the design target problem in problem (1) can be stated as follows: Determine the values of design variables $\boldsymbol{x}$ that minimize the deviation of supersystem responses $\boldsymbol{R}$ from predefined targets $\boldsymbol{T}$ subject to design constraints. Under the assumption that supersystem responses depend on supersystem design variables and system responses, system responses depend on system design variables and subsystem responses, and so on, down the design hierarchy, it can be concluded that supersystem responses and constraints depend on supersystem, system, subsystem, and component design variables.

When a hierarchical structure of the design target problem with $N+1$ levels is assumed, problem (1) can be expressed as follows:

$$\min_{\{x_{ij}, y_{ij} \mid j \in \mathcal{E}_i, i=0,\ldots,N\}} \|\boldsymbol{R}_{0l} - \boldsymbol{T}\|, \qquad l \in \mathcal{E}_0$$

subject to

$$\boldsymbol{g}_{ij}(\boldsymbol{R}_{ij}, \boldsymbol{x}_{ij}, \boldsymbol{y}_{ij}) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}(\boldsymbol{R}_{ij}, \boldsymbol{x}_{ij}, \boldsymbol{y}_{ij}) = \boldsymbol{0}$$

$$\boldsymbol{R}_{ij} - \boldsymbol{r}_{ij}\left(\boldsymbol{R}_{(i+1)k_1}, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}, \boldsymbol{x}_{ij}, \boldsymbol{y}_{ij}\right) = \boldsymbol{0}$$

$$\forall j \in \mathcal{E}_i, \qquad i = 0, \ldots, N \qquad (2)$$

where, for each element $j$ at the $i$th level, the following hold:

1) The vector of $n_{ij}$ local design variables is $\boldsymbol{x}_{ij} \in \mathbb{R}^{n_{ij}}$, that is, variables exclusively associated with the element.

2) The vector of $l_{ij}$ linking design variables is $\boldsymbol{y}_{ij} \in \mathbb{R}^{l_{ij}}$, that is, variables associated with the element and one or more other elements that share the same parent. Compatibility among linking design variables is enforced by sharing components of the vectors $\boldsymbol{y}_{ij}$ between different elements $j$ that share the same parent.

3) $\boldsymbol{R}_{ij} \in \mathbb{R}^{d_{ij}}$ is the vector of $d_{ij}$ responses.

4) Response vector functions include $\boldsymbol{r}_{ij}$: $\mathbb{R}^{a_{ij}} \to \mathbb{R}^{d_{ij}}$, where

$$a_{ij} := n_{ij} + l_{ij} + \sum_{k \in \mathcal{C}_{ij}} d_{(i+1)k}$$

5) Vector functions include $\boldsymbol{g}_{ij}$: $\mathbb{R}^{b_{ij}} \to \mathbb{R}^{v_{ij}}$ and $\boldsymbol{h}_{ij}$: $\mathbb{R}^{b_{ij}} \to \mathbb{R}^{u_{ij}}$, representing $v_{ij}$ inequality and $u_{ij}$ equality design constraints, respectively, where $b_{ij} := d_{ij} + n_{ij} + l_{ij}$.

6) $\|\cdot\|$ is a norm; typically, a weighted norm is used for the targets $\boldsymbol{T}$ to enable tradeoff evaluation studies, whereas the $l_2$-norm is used in all other cases.

Note that at the (top) zeroth level, there are no linking design variables $\boldsymbol{y}_{0l}$, $l \in \mathcal{E}_0$. Moreover, at the (bottom) $N$th level, element responses depend only on the element's design variables; thus, the last equality constraints in expression (2) become $\boldsymbol{R}_{Nj} - \boldsymbol{r}_{Nj}(\boldsymbol{x}_{Nj}, \boldsymbol{y}_{Nj}) = \boldsymbol{0}$, $\forall j \in \mathcal{E}_N$.

### C. Relaxation of the Design Target Problem

To study the convergence properties of ATC applied to the design target problem (2), we relax the interactions between every two consecutive levels of the problem hierarchy by introducing local copies of the responses at each level. Namely, we introduce superscripts that denote the levels at which the individual quantities are computed. This adds notational complexity but is necessary to indicate clearly the level at which the quantities are computed. Moreover, compatibility of linking design variables at a given level is enforced by introducing copies of these variables at the corresponding upper level. Hence, a relaxation of the problem in expression (2) is

$$\min_{\{\bar{x}_{ij}^i \mid j \in \mathcal{E}_i, i=0,\ldots,N\}} \left\|\boldsymbol{R}_{0l}^0 - \boldsymbol{T}\right\| + \sum_{i=0}^{N-1} \sum_{j \in \mathcal{E}_i} \epsilon_{ij}^R + \sum_{i=0}^{N-1} \sum_{j \in \mathcal{E}_i} \epsilon_{ij}^y, \qquad l \in \mathcal{E}_0$$

subject to

$$\boldsymbol{g}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\boldsymbol{R}_{ij}^i - \boldsymbol{r}_{ij}\left(\boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\forall j \in \mathcal{E}_i, \qquad i = 0, \ldots, N,$$

$$\sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^R \left\|\boldsymbol{R}_{(i+1)k}^i - \boldsymbol{R}_{(i+1)k}^{i+1}\right\| \le \epsilon_{ij}^R$$

$$\sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^y \left\|\boldsymbol{y}_{(i+1)k}^i - \boldsymbol{y}_{(i+1)k}^{i+1}\right\| \le \epsilon_{ij}^y$$

$$\forall j \in \mathcal{E}_i, \qquad i = 0, \ldots, N-1 \qquad (3)$$

In the preceding formulation for element $j$ at the $i$th level and element $k$ at the $(i+1)$th level, the following hold true:

1)

$$\bar{\boldsymbol{x}}_{ij}^i := \left[\boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i, \boldsymbol{y}_{(i+1)k_1}^i, \ldots, \boldsymbol{y}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \epsilon_{ij}^R, \epsilon_{ij}^y\right]^t$$

is the vector of optimization variables.

2) The weighting coefficient for the deviation of responses $\boldsymbol{R}_{(i+1)k}$ computed at the $i$th and $(i+1)$th levels is $w_{(i+1)k}^R \in \mathbb{R}_{\ge 0}$.

3) The tolerance variable for compatibility of responses between the $i$th and $(i+1)$th levels is $\epsilon_{ij}^R \in \mathbb{R}_{\ge 0}$.

4) The weighting coefficient for the deviation of linking design variables $\boldsymbol{y}_{(i+1)k}$ computed at the $i$th and $(i+1)$th level is $w_{(i+1)k}^y \in \mathbb{R}_{\ge 0}$.

5) The tolerance variable for compatibility of linking design variables at the $(i+1)$th level is $\epsilon_{ij}^y \in \mathbb{R}_{\ge 0}$.

6) $\boldsymbol{R}_{ij}^i \in \mathbb{R}^{d_{ij}}$ is the $i$th level copy of the vector of $d_{ij}$ responses.

7) $\boldsymbol{R}_{(i+1)k}^i \in \mathbb{R}^{l_{(i+1)k}}$, $k \in \mathcal{C}_{ij}$, is the $i$th level copy of the vector of $d_{(i+1)k}$ responses associated with the children of element $j$.

8) The same vector as $\boldsymbol{x}_{ij}$, the vector of $n_{ij}$ local design variables is $\boldsymbol{x}_{ij}^i \in \mathbb{R}^{n_{ij}}$. Although there is only one copy of these variables, the superscript $i$ has been added for consistency with the linking design variables.

9) The $i$th level copy of the vector of $l_{ij}$ linking design variables, that is, variables associated with the element and one or more other elements that share the same parent, is $\boldsymbol{y}_{ij}^i \in \mathbb{R}^{l_{ij}}$. Note that $\boldsymbol{y}_{ij}^i$ for $j \in \mathcal{E}_i$ are independent of each other, that is, they do not share components.

10) The $i$th level copy of the vector of $l_{(i+1)k}$ linking design variables associated with the children of element $j$ is $\boldsymbol{y}_{(i+1)k}^i \in \mathbb{R}^{l_{(i+1)k}}$, $k \in \mathcal{C}_{ij}$. Compatibility among linking design variables of the children of element $j$ is enforced by sharing components of the vectors $\boldsymbol{y}_{(i+1)k}^i$ between different elements $k \in \mathcal{C}_{ij}$.

Note again that at the (top) zeroth level, there are no linking design variables $\boldsymbol{y}_{0l}^0$, $l \in \mathcal{E}_0$. Moreover, at the (bottom) $N$th level, element responses depend only on the element's design variables; thus, the last equality constraints expression (3) become $\boldsymbol{R}^N_{Nj} - \boldsymbol{r}_{Nj}(\boldsymbol{x}_{Nj}, \boldsymbol{y}^N_{Nj}) = \boldsymbol{0}$, for $\forall j \in \mathcal{E}_N$. Also, $\epsilon^R_{ij}$ and $\epsilon^y_{ij}$ appear in the objective function to be minimized and, thus, the inequality constraints involving $\epsilon^R_{ij}$ and $\epsilon^y_{ij}$ are active and can be included in the objective function, as follows:

$$\min_{\{\bar{x}^i_{ij} \mid j \in \mathcal{E}_i, i = 0,\ldots,N\}} \left\|\boldsymbol{R}^0_{0l} - \boldsymbol{T}\right\| + \sum_{i=0}^{N-1} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w^R_{(i+1)k} \left\|\boldsymbol{R}^i_{(i+1)k} - \boldsymbol{R}^{i+1}_{(i+1)k}\right\| + \sum_{i=0}^{N-1} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w^y_{(i+1)k} \left\|\boldsymbol{y}^i_{(i+1)k} - \boldsymbol{y}^{i+1}_{(i+1)k}\right\|, \qquad l \in \mathcal{E}_0$$

subject to

$$\boldsymbol{g}_{ij}\left(\boldsymbol{R}^i_{ij}, \boldsymbol{x}^i_{ij}, \boldsymbol{y}^i_{ij}\right) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}\left(\boldsymbol{R}^i_{ij}, \boldsymbol{x}^i_{ij}, \boldsymbol{y}^i_{ij}\right) = \boldsymbol{0}$$

$$\boldsymbol{R}^i_{ij} - \boldsymbol{r}_{ij}\left(\boldsymbol{R}^i_{(i+1)k_1}, \ldots, \boldsymbol{R}^i_{(i+1)k_{c_{ij}}}, \boldsymbol{x}^i_{ij}, \boldsymbol{y}^i_{ij}\right) = \boldsymbol{0}$$

$$\forall j \in \mathcal{E}_i, \qquad i = 0, \ldots, N \qquad (4)$$

where

$$\bar{\boldsymbol{x}}^i_{ij} := \left[\boldsymbol{x}^i_{ij}, \boldsymbol{y}^i_{ij}, \boldsymbol{y}^i_{(i+1)k_1}, \ldots, \boldsymbol{y}^i_{(i+1)k_{c_{ij}}}, \boldsymbol{R}^i_{(i+1)k_1}, \ldots, \boldsymbol{R}^i_{(i+1)k_{c_{ij}}}\right]^t$$

is the vector of optimization variables of element $j$ at the $i$th level.

### D. Description of the ATC Process

The ATC process solves a series of design target subproblems for each element in the design hierarchy, as shown in Fig. 3 for a hypothetical four-level hierarchy. The design target subproblem at the supersystem level is to minimize the deviation between supersystem targets and associated responses subject to constraints for system responses, system linking design variables, and supersystem design constraints. Once system responses and linking design variables are determined by solving this subproblem, they are cascaded down to the system level as targets. Similar subproblems are formulated for each element at lower levels of the hierarchy. Optimal responses and linking design variables are also passed up as constraint parameters to upper-level subproblems. Figure 4 shows the information flow described in term of responses, local, and linking design variables for a bilevel hierarchy. The design subproblems, which are represented by rectangular boxes, are solved according to the formulation given in Sec. II.E, problem (5). Analysis models, which are represented by ovals, are used to compute responses according to $\boldsymbol{R}_{ij} = \boldsymbol{r}_{ij}(\boldsymbol{R}_{(i+1)k_1}, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}, \boldsymbol{x}_{ij}, \boldsymbol{y}_{ij})$.

The top-down and bottom-up, level-by-level sequential solution of design subproblems just described and depicted in Figs. 3 and 4 is not the only coordination sequence allowable in ATC. Other solution sequences can also be used to implement a convergent ATC process. As shown later in this paper, an acceptable solution sequence recursively divides the problem hierarchy into two subforests until the resulting subforests correspond to single levels of the original hierarchy. The separable subproblems for the elements of these single-level forests are then solved according to the general ATC formulation given in Sec. II.E, problems (5) and (6). Figure 5 shows four convergent solution sequences that can be used to implement ATC for a four-level design hierarchy. In Fig. 5a, the supersystem subproblem is solved first, then the system subproblem is solved, then subsystem and component subproblems are solved iteratively until

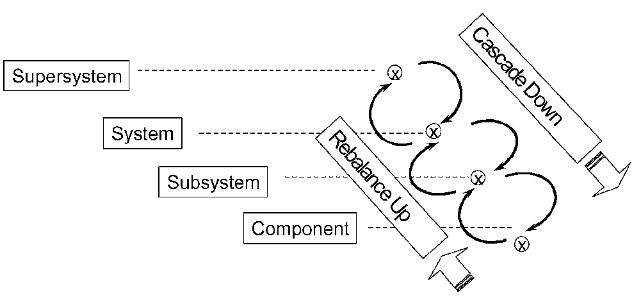


**Fig. 3 Schematic of the analytical target cascading process for a top-down and bottom-up, level-by-level solution sequence.**

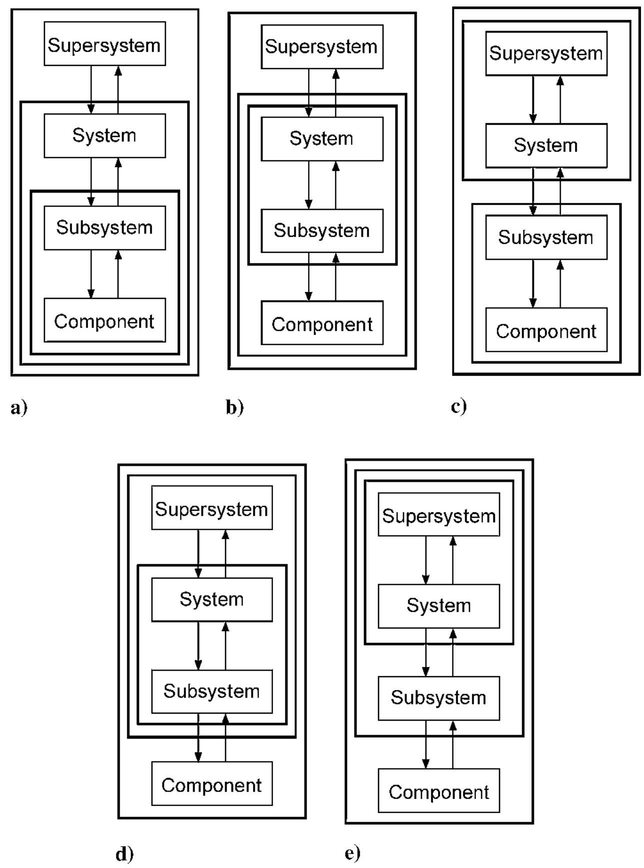


**Fig. 5 Schematics of analytical target cascading processes for various convergent solution sequences.**

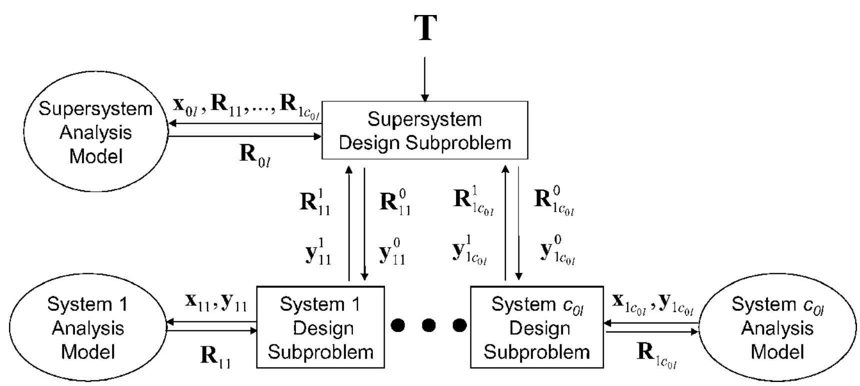


**Fig. 4 Information flow in analytical target cascading for a bilevel hierarchy.**

some convergence criterion is achieved. This loop is then expanded to include system subproblems and later the supersystem subproblem. In Fig. 5b, the supersystem subproblem is solved first, and then system and subsystem subproblems are solved iteratively until convergence. This loop is expanded to include component subproblems and later the supersystem subproblem. In Fig. 5c, supersystem and system subproblems are solved iteratively until convergence. Then subsystem and component subproblems are solved iteratively until convergence. Iteration continues between the supersystem/systems loop and the subsystems/components loop. The iterations in Fig. 5d are analogous to those in Fig. 5b after swapping supersystem and components and systems and subsystems in the solution sequence. The same holds true for the iteration sequence in Fig. 5e with respect to that in Fig. 5a.

### E. Formulation of ATC at a Given Level of the Hierarchy

The ATC design subproblem corresponding to the element $j$ at the $i$th (intermediate) level is formulated as follows:

$$\min_{\bar{x}_{ij}} w_{ij}^R \left\| \boldsymbol{R}_{ij}^i - \boldsymbol{R}_{ij}^{i-1} \right\| + w_{ij}^y \left\| \boldsymbol{y}_{ij}^i - \boldsymbol{y}_{ij}^{i-1} \right\| + \epsilon_{ij}^R + \epsilon_{ij}^y$$

subject to

$$\sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^R \left\| \boldsymbol{R}_{(i+1)k}^i - \boldsymbol{R}_{(i+1)k}^{i+1} \right\| \le \epsilon_{ij}^R$$

$$\sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^y \left\| \boldsymbol{y}_{(i+1)k}^i - \boldsymbol{y}_{(i+1)k}^{i+1} \right\| \le \epsilon_{ij}^y$$

$$\boldsymbol{g}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\boldsymbol{R}_{ij}^i - \boldsymbol{r}_{ij}\left(\boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0} \tag{5}$$

In the preceding formulation for the element $j$ at the $i$th level,

$$\bar{\boldsymbol{x}}_{ij} := \left[\boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i, \boldsymbol{y}_{(i+1)k_1}^i, \ldots, \boldsymbol{y}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \epsilon_{ij}^R, \epsilon_{ij}^y\right]^t$$

is the vector of optimization variables.

At the top level, the term $\boldsymbol{R}_{0l}^{-1}$, $l \in \mathcal{E}_0$, can be considered as the vector of supersystem targets $\boldsymbol{T}$, and the term $\|\boldsymbol{y}_{0j} - \boldsymbol{y}_{0j}^{-1}\|$ is missing. At the bottom level, the deviation constraints on responses and linking design variables are missing.

It is assumed for simplicity that all problems are continuous, but the formulation holds even if some optimization variables are discrete. In the latter case, suitable optimization algorithms are necessary for the solution of the associated mixed-integer programming problems.

Note again that $\epsilon_{ij}^R$ and $\epsilon_{ij}^y$ appear in the objective function to be minimized, and, thus, the two inequality constraints involving $\epsilon_{ij}^R$ and $\epsilon_{ij}^y$ are active and can be included in the objective function. Moreover, the subproblems in expression (5) for the elements at level $i$ are independent of each other and can be combined into the following single problem by taking the summation of the objective functions and the union of the constraint functions:

$$\min_{\{\bar{x}_{ij}^i \mid j \in \mathcal{E}_i\}} \sum_{j \in \mathcal{E}_i} w_{ij}^R \left\| \boldsymbol{R}_{ij}^i - \boldsymbol{R}_{ij}^{i-1} \right\| + \sum_{j \in \mathcal{E}_i} w_{ij}^y \left\| \boldsymbol{y}_{ij}^i - \boldsymbol{y}_{ij}^{i-1} \right\|$$
$$+ \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^R \left\| \boldsymbol{R}_{(i+1)k}^i - \boldsymbol{R}_{(i+1)k}^{i+1} \right\|$$
$$+ \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^y \left\| \boldsymbol{y}_{(i+1)k}^i - \boldsymbol{y}_{(i+1)k}^{i+1} \right\|$$

subject to

$$\boldsymbol{g}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\boldsymbol{R}_{ij}^i - \boldsymbol{r}_{ij}\left(\boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\forall j \in \mathcal{E}_i \tag{6}$$

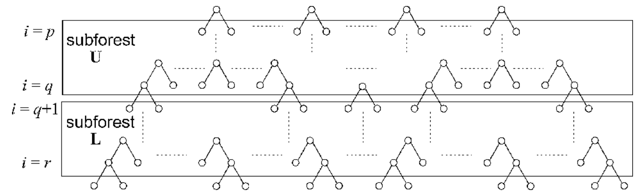


**Fig. 6 General forest in the problem hierarchy covering all nodes and edges from level $i = p$ to $r$**

The ATC process solves the subproblems in expression (6) for all levels of the design hierarchy in an orderly and iterative fashion, converging to the solution of the original design target problem (1) or (2). Some convergent solution sequences were shown in Fig. 5. The main outcome of the ATC process is the final values of element responses, which represent the (cascaded) targets for the systems, subsystems, and components of the supersystem.

## III. Design Target Subproblem for a Forest in the Problem Hierarchy

In this section we define carefully the structure of the two adjacent subproblems that must be jointly solved iteratively in any ATC convergent solution strategy, such as those depicted in Fig. 5. This will prepare us for the convergence arguments in Secs. IV and V.

Consider a forest $\boldsymbol{F}$ in the problem hierarchy covering all nodes and edges from level $i = p$ to $r \ge p + 1$. Figure 6 depicts such a general forest.

Let $q$ be a number between $p$ and $r$. Decompose the forest $\boldsymbol{F}$ into two subforests $\boldsymbol{U}$ and $\boldsymbol{L}$. The subforest $\boldsymbol{U}$ consists of all of the levels between $i = p$ and $q$, whereas the subforest $\boldsymbol{L}$ consists of all of the levels between $i = q + 1$ and $r$.

Then, from the overall problem expression (4), the relaxed design target subproblem corresponding to the forest $\boldsymbol{F}$ can be formulated as in expression (7). This subproblem consists of all design constraints and responses between levels $p$ and $r$. Given that the subproblems for the subtrees rooted at the nodes at level $i = p$, that is, elements of $\mathcal{E}_p$, are independent of each other, they can be combined into a single problem by taking the summation of their objective functions and the union of their constraint functions:

$$\min_{\{\bar{x}_{ij}^i \mid j \in \mathcal{E}_i, i = p, \ldots, r\}} \sum_{i=p-1}^{r} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^R \left\| \boldsymbol{R}_{(i+1)k}^i - \boldsymbol{R}_{(i+1)k}^{i+1} \right\|$$
$$+ \sum_{i=p-1}^{r} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^y \left\| \boldsymbol{y}_{(i+1)k}^i - \boldsymbol{y}_{(i+1)k}^{i+1} \right\|$$

subject to

$$\boldsymbol{g}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) \le \boldsymbol{0}, \qquad \boldsymbol{h}_{ij}\left(\boldsymbol{R}_{ij}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\boldsymbol{R}_{ij}^i - \boldsymbol{r}_{ij}\left(\boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i\right) = \boldsymbol{0}$$

$$\forall j \in \mathcal{E}_i, \qquad p \le i \le r \tag{7}$$

where

$$\bar{\boldsymbol{x}}_{ij}^i := \left[\boldsymbol{x}_{ij}^i, \boldsymbol{y}_{ij}^i, \boldsymbol{y}_{(i+1)k_1}^i, \ldots, \boldsymbol{y}_{(i+1)k_{c_{ij}}}^i, \boldsymbol{R}_{(i+1)k_1}^i, \ldots, \boldsymbol{R}_{(i+1)k_{c_{ij}}}^i\right]^t$$

is the vector of optimization variables of node $j$ at level $i$. Responses and linking design variables with superscripts $p - 1$ and $r + 1$ are fixed parameters cascaded down and passed up from upper and lower levels, respectively.

### A. Problem for the Upper Subforest

The design target subproblem for the upper subforest $\boldsymbol{U}$ can be derived from the problem for $\boldsymbol{F}$ in expression (7) simply by replacing $r$ by $q$:

$$\min_{\{\bar{x}_{ij}^i \mid j \in \mathcal{E}_i, i = p, \ldots, q\}} \sum_{i=p-1}^{q} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^R \left\| \boldsymbol{R}_{(i+1)k}^i - \boldsymbol{R}_{(i+1)k}^{i+1} \right\|$$
$$+ \sum_{i=p-1}^{q} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w_{(i+1)k}^y \left\| \boldsymbol{y}_{(i+1)k}^i - \boldsymbol{y}_{(i+1)k}^{i+1} \right\|$$

subject to

$$g_{ij}\left(R^i_{ij}, x^i_{ij}, y^i_{ij}\right) \le 0, \qquad h_{ij}\left(R^i_{ij}, x^i_{ij}, y^i_{ij}\right) = 0$$

$$R^i_{ij} - r_{ij}\left(R^i_{(i+1)k_1}, \ldots, R^i_{(i+1)k_{c_{ij}}}, x^i_{ij}, y^i_{ij}\right) = 0$$

$$\forall j \in \mathcal{E}_i, \qquad p \le i \le q \quad (8)$$

where

$$\bar{x}^i_{ij} := \left[x^i_{ij}, y^i_{ij}, y^i_{(i+1)k_1}, \ldots, y^i_{(i+1)k_{c_{ij}}}, R^i_{(i+1)k_1}, \ldots, R^i_{(i+1)k_{c_{ij}}}\right]^t$$

is the vector of optimization variables of node $j$ at level $i$. Responses and linking design variables with $p-1$ and $q+1$ superscripts are fixed parameters cascaded down and passed up from upper and lower levels, respectively.

### B. Problem for the Lower Subforest

The design target subproblem for the lower subforest $\boldsymbol{L}$ can be derived from the problem for $\boldsymbol{F}$ in expression (7) simply by replacing $p-1$ by $q$:

$$\min_{\{\bar{x}^i_{ij} \mid j \in \mathcal{E}_i, i = q+1,\ldots,r\}} \sum_{i=q}^{r} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w^R_{(i+1)k} \left\| R^i_{(i+1)k} - R^{i+1}_{(i+1)k} \right\|$$

$$+ \sum_{i=q}^{r} \sum_{j \in \mathcal{E}_i} \sum_{k \in \mathcal{C}_{ij}} w^y_{(i+1)k} \left\| y^i_{(i+1)k} - y^{i+1}_{(i+1)k} \right\|$$

subject to

$$g_{ij}\left(R^i_{ij}, x^i_{ij}, y^i_{ij}\right) \le 0, \qquad h_{ij}\left(R^i_{ij}, x^i_{ij}, y^i_{ij}\right) = 0$$

$$R^i_{ij} - r_{ij}\left(R^i_{(i+1)k_1}, \ldots, R^i_{(i+1)k_{c_{ij}}}, x^i_{ij}, y^i_{ij}\right) = 0$$

$$\forall j \in \mathcal{E}_i, \qquad q+1 \le i \le r \quad (9)$$

where

$$\bar{x}^i_{ij} := \left[x^i_{ij}, y^i_{ij}, y^i_{(i+1)k_1}, \ldots, y^i_{(i+1)k_{c_{ij}}}, R^i_{(i+1)k_1}, \ldots, R^i_{(i+1)k_{c_{ij}}}\right]^t$$

is the vector of optimization variables of node $j$ at level $i$. Responses and linking design variables with $q$ and $r+1$ superscripts are fixed parameters cascaded down and passed up from upper and lower levels, respectively.

## IV. ATC as Problem Coordination Between Two Subforests

The problem for the forest $\boldsymbol{F}$ in expression (7) can be rewritten in the following simplified form:

$$\min_x f(x) \qquad \text{subject to} \qquad g(x) \le 0, \qquad h(x) = 0 \quad (10)$$

where $x \in \mathbb{R}^n$ is the vector of all (independent and dependent) variables in problem (7). Let $x_U \in \mathbb{R}^{n_U}$ be the vector of all variables present in the upper subforest problem (8) and $x_L \in \mathbb{R}^{n_L}$ be the vector of all variables present in the lower subforest problem in (9), where $n = n_U + n_L$. Thus, by reordering variables if necessary, we have

$$x := \begin{pmatrix} x_U \\ x_L \end{pmatrix} \quad (11)$$

Note that $x_U$ consists of variables with superscripts between $p$ and $q$, and $x_L$ consists of variables with superscripts between $q+1$ and $r+1$.

Define $H_U$ to be the submatrix of the identity matrix $I_n$ consisting of its first $n_U$ rows and $H_L$ to be the submatrix of $I_n$ consisting of its last $n_L$ rows. Then,

$$I_n = \begin{pmatrix} H_U \\ H_L \end{pmatrix}, \qquad H_U x = x_U, \qquad H_L x = x_L \quad (12)$$

The upper subforest problem (8) and the lower subforest problem (9) can be recovered by applying the HOC process[17,18] to the combined problem (7). That is, the problem in (8) can be recovered from the combined problem (7) by fixing the quantities with superscripts between $q+1$ and $r+1$ as constants, whereas problem (9) can be recovered from the combined problem (7) by fixing quantities with superscripts between $p-1$ and $q$ as constants. Moreover, quantities fixed to recover the upper subforest and lower subforest problems from the combined problem are determined from the solutions of the lower subforest and upper subforest problems, respectively.

The preceding coordination process, which corresponds to ATC applied to a forest $\boldsymbol{F}$ consisting of two subforests $\boldsymbol{U}$ and $\boldsymbol{L}$, can be rephrased using the simplified notation in problems (10) and (11): The variables $x_U$ in the upper subforest are first fixed at some feasible values, and then problem (10) is solved. This determines the values of $x_L$. Problem (10) is solved again with these fixed values of $x_L$, which determines new values of $x_U$. This iterative process is repeated until a stable set of variable values are obtained, that is, until the process converges.

One can easily prove that the preceding process converges[17,18] because, as shown in Sec. V.A, the values of the objective function in problem (10) are decreasing during the iteration, and a monotonically decreasing sequence bounded below always converges. However, it is not certain that the accumulation point obtained by this process corresponds to the optimal solution of the combined problem (10). This convergence will be addressed in Sec. V.B.

The "passing-down" and "passing-up" ATC process can also be described in terms of the matrices $H_U$ and $H_L$ in problem (12). That is, for fixed and feasible values of the variables in the upper subforest, $x_U = d_U$, the passing-down problem is

$$\min_x f(x) \quad \text{subject to} \quad g(x) \le 0, \quad h(x) = 0, \quad H_U x = d_U \quad (13)$$

whereas for fixed and feasible values of the variables in the lower subforest, $x_L = d_L$, the passing-up problem is

$$\min_x f(x) \quad \text{subject to} \quad g(x) \le 0, \quad h(x) = 0, \quad H_L x = d_L \quad (14)$$

As stated earlier in this section, $d_U$ ($d_L$, respectively) is updated during the iterative ATC process by solving problem (14) [problem (13), respectively].

The following lemma states that the constraint vectors $g$ and $h$ in problem (10) have a separable structure.

*Lemma:* The constraint vectors $g \in \mathbb{R}^{p^I}$ and $h \in \mathbb{R}^{p^E}$ in problem (10), by reordering the constraints if necessary, have the following separable structure:

$$g = \begin{pmatrix} g_U \\ g_L \end{pmatrix}, \qquad h = \begin{pmatrix} h_U \\ h_L \end{pmatrix}$$

where $g_U \in \mathbb{R}^{p^I_U}$ and $h_U \in \mathbb{R}^{p^E_U}$ ($g_L$ and $h_L$, respectively) contain only variables $x_U$ ($x_L$, respectively).

*Proof:* Define $g_U$, $g_L$, $h_U$, and $h_L$ as follows:

$$g_U := [\{g_{ij} \mid j \in \mathcal{E}_i, p \le i \le q\}]^t$$

$$g_L := [\{g_{ij} \mid j \in \mathcal{E}_i, q+1 \le i \le r\}]^t$$

$$\mathbf{h}_U := \left[\left\{h_{ij}, R^i_{ij} - r_{ij} \mid j \in \mathcal{E}_i, p \le i \le q\right\}\right]^t$$

$$h_L := \left[\left\{h_{ij}, R^i_{ij} - r_{ij} \mid j \in \mathcal{E}_i, q+1 \le i \le r\right\}\right]^t$$

Dependency of constraint functions $g_U$ and $h_U$ ($g_L$ and $h_L$, respectively) on variables $x_U$ ($x_L$, respectively) follows from constraint definitions in expressions (8) and (9). That is, constraint functions $g_U$ and $h_U$ ($g_L$ and $h_L$, respectively) depend on responses $R^i_{ij}$, local design variables $x^i_{ij}$ and linking design variables $y^i_{ij}$ with superscripts $i$ between $p$ and $q$ ($q+1$ and $r$, respectively). □

The separable structure of the constraints in expression (7) plays a crucial role in proving the convergence of ATC because the objective function is also separable with respect to $x_U$ and $x_L$. That is, the solutions for problems (8) and (9) can be recovered from the solution for problem (7).

*Remark:* The integer $p^I$ ($p^E$, respectively) indicates the number of inequality (equality, respectively) constraints in problem (10). The integer $p_U^I$ ($p_U^E$, respectively) is the number of inequality (equality, respectively) constraints involving variables $\boldsymbol{x}_U$ only.

## V. Convergence of ATC

We are now ready to complete the convergence proof for the ATC coordination process. Essentially, we justify why the coordination paths illustrated earlier in Fig. 5 will actually lead to the solution of the original design target problem.

### A. Convergence to an Accumulation Point

As mentioned in Sec. III, the ATC process applied to a general forest $\boldsymbol{F}$ of the problem hierarchy corresponds to the HOC process.[17,18,20] The following properties have been observed previously[20] and show convergence of the ATC process to an accumulation point:

1) If the HOC algorithm is initiated with a feasible point $\boldsymbol{x}_0$, then at each stage of the process, problems (13) and (14) will have nonempty feasible domains. Note that a feasible point $\boldsymbol{x}_0$ could be found by minimizing constraint violation.

2) If the sequences $\{\boldsymbol{x}_{L_i}\}_{i=1}^{\infty}$ and $\{\boldsymbol{x}_{U_i}\}_{i=1}^{\infty}$ result from solving problem (13) and problem (14), respectively, and $f^{\min} := \min\{f(\boldsymbol{x}) \mid \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\}$, then a) $f(\boldsymbol{x}_{U_i}, \boldsymbol{x}_{L_i}) \geq f(\boldsymbol{x}_{U_i}, \boldsymbol{x}_{L(i+1)}) \geq f(\boldsymbol{x}_{U(i+1)}, \boldsymbol{x}_{L(i+1)})$ and b) $\lim_{i\to\infty} f(\boldsymbol{x}_{L_i}) = \lim_{i\to\infty} f(\boldsymbol{x}_{U_i}) = f^* \geq f^{\min}$.

3) Any accumulation point $\boldsymbol{x}^*$ of either $\{\boldsymbol{x}_{L_i}\}_{i=1}^{\infty}$ or $\{\boldsymbol{x}_{U_i}\}_{i=1}^{\infty}$ solves both problems (13) and (14).

### B. Convergence to the Optimum of Overall Problem

Let $J(\boldsymbol{x})$ be the $(p^I + p^E) \times n$ matrix

$$\begin{pmatrix} J^I(\boldsymbol{x}) \\ J^E(\boldsymbol{x}) \end{pmatrix}$$

where $J^I(\boldsymbol{x})$ and $J^E(\boldsymbol{x})$ are the Jacobians of $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$, respectively. This matrix function $J(\boldsymbol{x})$ will be simply referred to as the Jacobian of problem (10).

For a fixed point $\boldsymbol{x}^* \in \mathbb{R}^n$, define $T_a(\boldsymbol{x}^*)$ to be the set of the indices corresponding to the active inequality constraints at $\boldsymbol{x}^*$, that is,

$$T_a(\boldsymbol{x}^*) := \left\{ i \;\middle|\; g_i(\boldsymbol{x}^*) = 0 \right\}$$

where $g_i$ denotes the $i$th inequality constraint.

Define an integer $p_a^I$ to be the number of active inequality constraints at $\boldsymbol{x}^*$. The integer $p_{a,U}^I$ ($p_{a,L}^I$, respectively) is the number of active inequality constraints involving variables $\boldsymbol{x}_U$ ($\boldsymbol{x}_L$, respectively) only. Note that the lemma in Sec. IV.A implies that $p_a^I = p_{a,U}^I + p_{a,L}^I$. Let a $p_a^I \times n$ matrix $J_a^I(\boldsymbol{x}^*)$ be the submatrix of $J^I(\boldsymbol{x}^*)$ consisting of the active inequality constraints at $\boldsymbol{x}^*$.

The Lagrange multiplier theorem[21] states that a regular point $\boldsymbol{x}^* \in \mathbb{R}^n$ is a solution to problem (10) if and only if there exists a nonnegative vector $\boldsymbol{\lambda}^I \in \mathbb{R}^{p_a^I}$ and a vector $\boldsymbol{\lambda}^E \in \mathbb{R}^{p^E}$ such that

$$\nabla f^t(\boldsymbol{x}^*) + J_a^I(\boldsymbol{x}^*)^t \boldsymbol{\lambda}^I + J^E(\boldsymbol{x}^*)^t \boldsymbol{\lambda}^E = \boldsymbol{0} \tag{15}$$

Condition (15) is equivalent to

$$-\nabla f^t(\boldsymbol{x}^*) = J_a^I(\boldsymbol{x}^*)^t \boldsymbol{\lambda}^I + J^E(\boldsymbol{x}^*)^t \boldsymbol{\lambda}^E, \qquad \boldsymbol{\lambda}^I \geq \boldsymbol{0} \tag{16}$$

Let $\boldsymbol{x}^*$ be an accumulation point of the ATC process for a general forest $\boldsymbol{F}$. Because it is a solution of both problems (13) and (14), there exist vectors $\boldsymbol{z}^I \geq \boldsymbol{0}$, $\boldsymbol{z}^E$, and $\boldsymbol{u}$ and vectors $\boldsymbol{w}^I \geq \boldsymbol{0}$, $\boldsymbol{w}^E$, and $\boldsymbol{v}$ such that the following two equalities simultaneously hold:

$$\begin{aligned} -\nabla f^t(\boldsymbol{x}^*) &= J_a^I(\boldsymbol{x}^*)^t \boldsymbol{z}^I + J^E(\boldsymbol{x}^*)^t \boldsymbol{z}^E + H_U^t \boldsymbol{u} \\ -\nabla f^t(\boldsymbol{x}^*) &= J_a^I(\boldsymbol{x}^*)^t \boldsymbol{w}^I + J^E(\boldsymbol{x}^*)^t \boldsymbol{w}^E + H_L^t \boldsymbol{v} \end{aligned} \tag{17}$$

Therefore, the convergence of the ATC process for a general forest $\boldsymbol{F}$ boils down to the following question: If $\boldsymbol{x}^*$ satisfies both Eqs. (17), does it automatically satisfy Eq. (16) for some vectors $\boldsymbol{\lambda}^I \geq \boldsymbol{0}$ and $\boldsymbol{\lambda}^E$?

The answer to this question is "yes," mainly due to the separable structure of the constraint functions shown in the lemma in Sec. IV.A.

*Theorem:* If $\boldsymbol{x}^*$ is a solution to both problems (13) and (14), then it is a solution to problem (10). That is, if $\boldsymbol{x}^*$ is a solution to both problems (8) and (9), then it is a solution to problem (7).

*Proof:* Let $\boldsymbol{x}^*$ be a solution to both problems (13) and (14). Then, there exist vectors

$$\boldsymbol{z}^I = \begin{pmatrix} z_1^I \\ \vdots \\ z_{p_a^I}^I \end{pmatrix} \geq \boldsymbol{0}, \qquad \boldsymbol{z}^E = \begin{pmatrix} z_1^E \\ \vdots \\ z_{p^E}^E \end{pmatrix}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{n_U} \end{pmatrix}$$

and the vectors

$$\boldsymbol{w}^I = \begin{pmatrix} w_1^I \\ \vdots \\ w_{p_a^I}^I \end{pmatrix} \geq \boldsymbol{0}, \qquad \boldsymbol{w}^E = \begin{pmatrix} w_1^E \\ \vdots \\ w_{p^E}^E \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{n_L} \end{pmatrix}$$

such that both Eqs. (17) simultaneously hold. Define a nonnegative vector $\boldsymbol{\lambda}^I \in \mathbb{R}^{p_a^I}$ and a vector $\boldsymbol{\lambda}^E \in \mathbb{R}^{p^E}$ as follows:

$$\boldsymbol{\lambda}^I = \begin{pmatrix} w_1^I \\ \vdots \\ w_{p_{a,U}^I}^I \\ z_{p_{a,U}}^I + 1^I \\ \vdots \\ z_{p_a^I}^I \end{pmatrix}, \qquad \boldsymbol{\lambda}^E = \begin{pmatrix} w_1^E \\ \vdots \\ w_{p_U^E}^E \\ z_{p_U^E+1}^E \\ \vdots \\ z_{p^E}^E \end{pmatrix} \tag{18}$$

Recall that the integer $p_{a,U}^I$ ($p_U^E$, respectively) is the number of active inequality constraints (equality constraints, respectively) involving variables $\boldsymbol{x}_U$ only. We claim that these $\boldsymbol{\lambda}^I$ and $\boldsymbol{\lambda}^E$ satisfy Eq. (16), which implies that $\boldsymbol{x}^*$ is a solution to problem (10).

To verify the claim, define the matrices $A_U^I$, $A_L^I$, $A_U^E$, and $A_L^E$, as follows:

$$J_a^I(\boldsymbol{x}^*) = \left(A_U^I, A_L^I\right), \qquad J^E(\boldsymbol{x}^*) = \left(A_U^E, A_L^E\right)$$

where $A_U^I$ and $A_U^E$ are the first $n_U$ columns of the matrices $J_a^I(\boldsymbol{x}^*)$ and $J^E(\boldsymbol{x}^*)$, respectively.

The two Eqs. (17) can be rewritten as

$$-\nabla f^t(\boldsymbol{x}^*) = \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} \boldsymbol{z}^I + \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} \boldsymbol{z}^E + H_U^t \boldsymbol{u}$$

$$= \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} \boldsymbol{w}^I + \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} \boldsymbol{w}^E + H_L^t \boldsymbol{v}$$

Because

$$\begin{pmatrix} H_U \\ H_L \end{pmatrix} = I_n$$

one gets

$$\begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} (\boldsymbol{z}^I - \boldsymbol{w}^I) + \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} (\boldsymbol{z}^E - \boldsymbol{w}^E) = H_L^t \boldsymbol{v} - H_U^t \boldsymbol{u}$$

$$= \left(H_U^t, H_L^t\right) \begin{pmatrix} -\boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$$

Therefore,

$$u = -A_U^{I\,t}(z^I - w^I) - A_U^{E\,t}(z^E - w^E)$$
$$v = A_L^{I\,t}(z^I - w^I) + A_L^{E\,t}(z^E - w^E)$$

Hence,

$$
\begin{aligned}
-\nabla f^t(x^*) &= \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} z^I + \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} z^E + H_U^t u \\
&= \left(H_U^t, H_L^t\right) \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} z^I + \left(H_U^t, H_L^t\right) \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} z^E \\
&\quad - H_U^t\left[A_U^{I\,t}(z^I - w^I) + A_U^{E\,t}(z^E - w^E)\right] \\
&= H_L^t A_L^{I\,t} z^I + H_L^t A_L^{E\,t} z^E + H_U^t A_U^{I\,t} w^I + H_U^t A_U^{E\,t} w^E \\
&= \left(H_U^t, H_L^t\right) \begin{pmatrix} A_U^{I\,t} w^I \\ A_L^{I\,t} z^I \end{pmatrix} + \left(H_U^t, H_L^t\right) \begin{pmatrix} A_U^{E\,t} w^E \\ A_L^{E\,t} z^E \end{pmatrix} \\
&= \begin{pmatrix} A_U^{I\,t} w^I \\ A_L^{I\,t} z^I \end{pmatrix} + \begin{pmatrix} A_U^{E\,t} w^E \\ A_L^{E\,t} z^E \end{pmatrix}
\end{aligned}
\tag{19}
$$

By the theorem in Sec. IV.A, the matrices $J_a^I(x^*)$ and $J^E(x^*)$ have the following block structure:

$$J_a^I(x^*) = \left(A_U^I, A_L^I\right) = \begin{pmatrix} \hat{A}_U^I & 0 \\ 0 & \hat{A}_L^I \end{pmatrix}$$

$$J^E(x^*) = \left(A_U^E, A_L^E\right) = \begin{pmatrix} \hat{A}_U^E & 0 \\ 0 & \hat{A}_L^E \end{pmatrix}$$

This implies the block structure

$$A_U^I = \begin{pmatrix} \hat{A}_U^I \\ 0 \end{pmatrix}, \qquad A_L^I = \begin{pmatrix} 0 \\ \hat{A}_L^I \end{pmatrix}$$

$$A_U^E = \begin{pmatrix} \hat{A}_U^E \\ 0 \end{pmatrix}, \qquad A_L^E = \begin{pmatrix} 0 \\ \hat{A}_L^E \end{pmatrix}$$

Using this block structure and the definitions of $\lambda^I$ and $\lambda^E$ in Eq. (18), one checks easily that

$$\begin{pmatrix} A_U^{I\,t} w^I \\ A_L^{I\,t} z^I \end{pmatrix} = \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} \lambda^I, \qquad \begin{pmatrix} A_U^{E\,t} w^E \\ A_L^{E\,t} z^E \end{pmatrix} = \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} \lambda^E$$

Combining the preceding equalities with Eq. (19), one gets

$$
\begin{aligned}
-\nabla f^t(x^*) &= \begin{pmatrix} A_U^{I\,t} \\ A_L^{I\,t} \end{pmatrix} \lambda^I + \begin{pmatrix} A_U^{E\,t} \\ A_L^{E\,t} \end{pmatrix} \lambda^E \\
&= J_a^I(x^*)^t \lambda^I + J^E(x^*)^t \lambda^E, \qquad \lambda^I \geq \mathbf{0}
\end{aligned}
$$

which shows that $x^*$ is indeed a solution of problem (10).

### C. Convergence of ATC to the Optimum of the Original Design Target Problem

Consider a general forest $F$ in the problem hierarchy covering all nodes and edges from level $i = p$ to level $i = r$. Decompose $F$ into two subforests, and apply ATC coordination to $F$ using this structure. It was shown in the preceding section that the ATC coordination process produces the optimum solution of the relaxed design target problem for $F$. Note that each subforest can be further decomposed into smaller subforests, and the same ATC process can be recursively applied to those subforests.

When $p = 0$ and $r = N$, forest $F$ becomes the hierarchy of the relaxed design target problems (3) and (4). Given that consistency and feasibility are assumed for the original design target problem, it is possible to find weights $w_{(i+1)k}^R$ and $w_{(i+1)k}^y$ such that $\epsilon_{(i+1)k}^R$ and $\epsilon_{(i+1)k}^y$ in problem (3) converge to zero. To be more specific, the relaxed design target problem (4) can be seen as a multiobjective optimization problem, with one objective being the deviation of the responses from the targets of the supersystem, and the other objectives being lower-level response residuals and linking variable residuals. The original design target problem has a solution because it is a feasible problem, and, therefore, this multiobjective optimization problem does have a Pareto solution minimizing the first objective while making other objectives zero. By relating to the weighting method[22] for finding Pareto solutions, we can conclude that there should exist weights that correspond to zero values of the residuals, as illustrated in Fig. 7. Note, however, that a further study is needed to come up with a strategy for finding these weights $w_{(i+1)k}^R$ and $w_{(i+1)k}^y$.

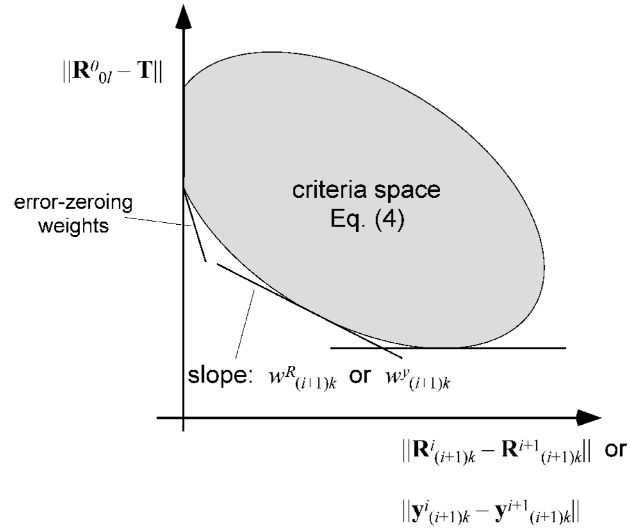


**Fig. 7 Existence of weights.**

This implies that the ATC process, recursively applied to the problem hierarchy, produces an optimum solution of the original design target problems (1) and (2).

## VI. Conclusions

The ATC problem formulation possesses a fortuitous structure that enables a convergent behavior of several coordination strategies, patterned after the earlier strategy of HOC. The main characteristic of the ATC convergent coordination strategies is the recursive solution of two overlapping problems at a time (Fig. 5). Our computational experiences to date with several realistic case studies cited in the introduction support these theoretical findings. Moreover, actual convergence has not presented any excessive computational burden. In fact, our computational experiences show that the convergence criteria for the inner loops in Fig. 5 can be initially relaxed and then tightened as the ATC process progresses. Complete relaxation of the inner-loop convergence tolerances results in the top-down and bottom-up, level-by-level solution sequence depicted in Fig. 3, which has shown to be convergent for most of our applications.

Although the implementations so far show fast convergence in a few iterations, and often show computational advantage, the real advantage is expected when the all-at-once approach is not available as an option. For example, in an organization where system design or component design is done by individual teams with their own design methodologies and cultures, the ATC process can be implemented with full expectation that the design outcome would be an optimal one.


## Acknowledgments

This work has been partially supported by the Automotive Research Center, a U. S. Army Center of Excellence for Modeling and Simulation of Ground Vehicles at the University of Michigan, and by a grant by Ford Motor Co. This support is gratefully acknowledged.

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