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COMPARISON OF COMBINED EMBODIMENT DESIGN AND CONTROL OPTIMIZATION STRATEGIES USING OPTIMALITY CONDITIONS

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ABSTRACT

Optimal embodiment design and control systems can be solved using several different strategies. This paper considers whether those strategies will find the true system optimum.

Optimality conditions determine whether a point is an optimum of a system. This paper presents a development and comparison of the optimality conditions for several methods of optimizing fully coupled embodiment design and control systems. The results demonstrate that the methods do not possess the same optimality conditions and therefore will not converge to the same optimal solutions, often leading to sub-optimal solutions. Since these strategies have been used successfully to solve various examples, criteria are developed to determine when the various methods will find the true system optimum.

NOMENCLATURE

Design Problem Formulation

$\boldsymbol{a}$	simple design parameters
$\boldsymbol{b}$	design parameters, based upon the *control* problem
$\boldsymbol{c}$	parameters of the design problem = $\{\boldsymbol{a}, \boldsymbol{b}\}$
$\boldsymbol{d}$	design variables
f	objective for the design problem
$\boldsymbol{h}$	equality constraints for the design problem
$\boldsymbol{g}$	inequality constraints for the design problem
q	number of design variables
$\boldsymbol{v}$	*control* parameters, based upon the design problem
$\boldsymbol{V}(...)$	coupling equations dependent on the design

Control Problem Formulation

$\boldsymbol{b}$	*design* parameters, based upon the control problem
$\boldsymbol{B}(...)$	coupling equations dependent on the control
J	performance index, objective for the control problem
j	time dependent performance index, integration yields J
$\boldsymbol{k}$	equality constraints for the control problem
$\boldsymbol{l}$	inequality constraints for the control problem
m	number of state variables
n	number of control variables
$\boldsymbol{p}$	vector of control gains for a controller configuration
$\boldsymbol{r}$	dynamical equations for the control problem
$\boldsymbol{s}$	response relations for the control problem
t	time continuum
$\boldsymbol{u}$	simple control parameters
$\boldsymbol{v}$	control parameters, based upon the *design* problem
$\boldsymbol{w}$	parameters of the control problem = $\{\boldsymbol{u}, \boldsymbol{v}\}$
$\boldsymbol{x}(t)$	state variables as a function of time
$\boldsymbol{y}(t)$	measurable response of the controlled system
$\boldsymbol{z}(t)$	control signal variables as a function of time
$\boldsymbol{Z}(...)$	control strategy equations
ζ	equality constraints relating the control signal and strategy

Combined Problem Formulation

β	coupling constraints: design inputs, control outputs
γ	coupling constraints: control inputs, design outputs
φ	system objective
ϖ_1, ϖ_2	objective weights

INTRODUCTION

Traditional embodiment design and control of an artifact or system has been performed in a sequential, if not altogether separate, manner. One creates a design and then designs a controller for that specific design. Optimization was naturally applied in the same sequence: optimize the design then optimize the control. The next logical improvement was to iteratively optimize the design then optimize the control, or combine them in some appropriate manner. Combined design and control has been studied for several systems, including structures and robotics [e.g., 1–3].

In structural design the first integration attempts were to optimize the traditional sequential analysis, namely, optimize around the entire sequence instead of around each part of the sequence. Another approach involved a bilevel optimization process: At each structural design point, a corresponding optimal controller was found using the analytical solution to a linear quadratic regulator (LQR) controller [4]. Of course, the most general approach is a true concurrent strategy, sometimes referred to as ‘all at once,’ where all objectives and constraints for both design and control are placed in a single optimization model and solved as one problem [5]. For many engineering problems, the complexity and size of the combined problem could make this concurrent strategy impractical, unless special decomposition techniques could be used.

In an earlier article, various strategies for optimizing combined design and control systems were studied in the context of a case study involving a direct current electric motor [6]. These strategies reached different results and did not necessarily lead to the best design and control solution. The reasons for such results are not readily apparent. One might suspect that unexpected results come from failure of the numerical optimization algorithms or from the presence of multiple optima.

In a sequel article, a simple demonstration example was used to trace the reasons for the discrepancies [7]. These discrepancies were found to depend on the correct identification of the activity of the coupling constraints, as well as to the actual formulation of the combined problem implied by each alternate strategy. The results demonstrated that the combined design and control problem presents significant challenges even in its simplest form, so system design should be conducted with an appropriate degree of caution.

In this article we once again visit the various solution strategies for optimizing combined embodiment design and control problems. We will formulate the optimality conditions for each strategy. From the conditions we will determine general circumstances under which the strategies can find the true optimum of the system.

DESIGN AND CONTROL SYSTEM MODELS

This section presents the three models that are considered in combined embodiment design and control: the design model, the control model and the combined model. The development of these models is explained in detail in Reyer [8].

Design Model

The design model is the standard optimal design formulation, Eq. (1).

$$\begin{aligned}
&\underset{\boldsymbol{d}\ \ \mathrm{D}}{\text{Minimize}}\ \ q \qquad f(\boldsymbol{d};\boldsymbol{c})\\
&\text{Subject to:}\\
&\qquad \boldsymbol{h}(\boldsymbol{d};\boldsymbol{c}) = \boldsymbol{0}\\
&\qquad \boldsymbol{g}(\boldsymbol{d};\boldsymbol{c}) \le \boldsymbol{0}\\
&\qquad \boldsymbol{c} = \{\boldsymbol{a},\boldsymbol{b}\}\\
&\text{Resulting in:}\\
&\qquad \boldsymbol{v} = V(\boldsymbol{d}^*,\boldsymbol{c})
\end{aligned} \tag{1}$$

In this model the objective f and constraints $\boldsymbol{g}$ and $\boldsymbol{h}$ depend on the variables $\boldsymbol{d}$ and the parameters $\boldsymbol{c}$. The parameters are divided into two groups: those that depend on the control results, $\boldsymbol{b}$, and those that do not, $\boldsymbol{a}$. The design variables are all static in that they do not change with time. A superscript * indicates optimal values.

Control Model

The control model is the standard optimal gains formulation, Eq. (2).

$$\begin{aligned}
&\underset{\boldsymbol{p}\qquad M}{\text{Minimize}}\qquad J = \int_0^{t_f} j(\boldsymbol{x}(t),\boldsymbol{z}(t),t;\boldsymbol{w})\mathrm{d}\boldsymbol{t}\\
&\text{Subject to:}\\
&\qquad \dot{\boldsymbol{x}} = \boldsymbol{r}(\boldsymbol{x}(t),\boldsymbol{z}(t),t;\boldsymbol{w})\\
&\qquad \boldsymbol{y} = \boldsymbol{s}(\boldsymbol{x}(t),\boldsymbol{z}(t),t;\boldsymbol{w})\\
&\qquad \boldsymbol{k}(\boldsymbol{x}(t),\boldsymbol{z}(t),t;\boldsymbol{w}) = 0\\
&\qquad \boldsymbol{l}(\boldsymbol{x}(t),\boldsymbol{z}(t),t;\boldsymbol{w}) \le 0\\
&\qquad \zeta = \boldsymbol{Z}(\boldsymbol{x}(t),\boldsymbol{p},t,\boldsymbol{v};\boldsymbol{u}) - \boldsymbol{z}(t) = 0\\
&\qquad \boldsymbol{w} = \{\boldsymbol{u},\boldsymbol{v}\}\\
&\text{Resulting in:}\\
&\qquad \boldsymbol{b} = \boldsymbol{B}(\boldsymbol{x}\ \ (t),\boldsymbol{z}\ \ (t),t\ \ ;\boldsymbol{w})
\end{aligned} \tag{2}$$

In this model a performance index J is minimized subject to the dynamical equations: the state equations $\boldsymbol{r}$, the output equations $\boldsymbol{s}$, and constraints $\boldsymbol{k}$ and $\boldsymbol{l}$. The ζ equation represents an a priori choice of a control strategy (e.g. state feedback) such that the time dependent input $\boldsymbol{z}$ depends on other available quantities and on time independent variable gains $\boldsymbol{p}$. The states $\boldsymbol{x}$, the input $\boldsymbol{z}$, and time t are intermediate variables in the problem. The parameters $\boldsymbol{w}$ are again split into two parts: those that depend on the design results, $\boldsymbol{v}$, and those that do not, $\boldsymbol{u}$.

Combined System Model

The combined system model appears in Eq. (3).

Minimize $\varphi = \varpi_1 f + \varpi_2 J$

With $f = f(\boldsymbol{d}, \boldsymbol{b}; \boldsymbol{a})$

$$J = \int_0^{t_f} j(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u})\mathrm{d}t$$

Variables: $\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{v}$

Internal Variables: t, $\boldsymbol{x}(t)$, $\boldsymbol{z}(t)$

Parameters: $\boldsymbol{a}, \boldsymbol{u}$

Subject to:

$$\begin{aligned}
&\boldsymbol{g}(\boldsymbol{d}, \boldsymbol{b}; \boldsymbol{a}) \le \boldsymbol{0} \\
&\boldsymbol{h}(\boldsymbol{d}, \boldsymbol{b}; \boldsymbol{a}) \le \boldsymbol{0} \\
&\boldsymbol{k}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) = \boldsymbol{0} \\
&\boldsymbol{l}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) \le \boldsymbol{0} \\
&\dot{\boldsymbol{x}} = \boldsymbol{r}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) \\
&\boldsymbol{y} = \boldsymbol{s}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) \\
&\beta^{\dagger} = \mathrm{B}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) - \boldsymbol{b} \le \boldsymbol{0} \\
&\gamma^{\dagger} = \boldsymbol{V}(\boldsymbol{d}, \boldsymbol{b}; \boldsymbol{a}) - \boldsymbol{v} \le \boldsymbol{0} \\
&\zeta = \boldsymbol{Z}(\boldsymbol{p}, \boldsymbol{x}(t), t, \boldsymbol{v}; \boldsymbol{u}) - \boldsymbol{z}(t) = \boldsymbol{0}
\end{aligned} \tag{3}$$

Note: † indicates that each element in the β and γ vectors has been directed accordingly

This model is an integration of the design and control models from Eqs. (1) and (2). The objective function φ is a weighted sum of the separate design and control objectives. The former parameters ***b*** and ***v*** are now variables, as the entire set of equations must be satisfied simultaneously. Coupling constraints β and γ enforce the relationship between the values of the coupling variables ***b*** and ***v*** and the resulting values from their corresponding models ***B*** and ***V***. Each element in the β and γ vectors must be properly directed.

Though comprehensive information on this coupling has been previously presented, see Reyer and Papalambros [7] and Reyer [8], the importance of the coupling constraints to the problem warrants further explanation. The impetus for using properly directed inequalities rather than equalities stems from the concept of *feasibility*. The feasible set of solutions are all of the solutions that satisfy the constraints in the problem. Consider common b and B values from an electric motor: the motor is designed with a specific power rating, b_m, and the controlled motor uses a specific amount of power, B_m. We recognize that if we use more power than the amount for which the motor was designed, the motor will fail. However, when less power is used than the amount for which the motor was designed, the motor will still work properly, though the motor may be inefficient. These ideas yield the inequality β_m: $B_m - b_m \le 0$. Forcing $\beta_m = 0$ will limit the feasible set and that, as in the case with an overpowered motor, the reduction of inequalities to equalities will not represent the physical condition. Rather, we will allow the optimization of the problem to determine when the coupling constraints should become strict equalities at the optimum.

A difficulty arises when we do not have a good physical understanding of a certain coupling constraint. Though in the motor β_m: $B_m - b_m \le 0$, another coupling constraint may need to be represented as β_i: $b_i - B_i \le 0$. Without an understanding of the physical problem, we cannot readily assign the direction of the coupling constraints. In the references cited above, a technique was developed for directing the coupling constraints. The direction of the coupling constraints can be determined by examining the problem with Monotonicity Analysis. However, each coupling constraint must be examined individually to determine its direction. The equations β: $\boldsymbol{B} - \boldsymbol{b} \le \boldsymbol{0}$ and γ: $\boldsymbol{V} - \boldsymbol{v} \le \boldsymbol{0}$, as used in this paper, are therefore representational.

This model is fully coupled. The design depends on the control and the control depends on the design. In an electric motor, for example, power and torque ratings are needed to design the motor windings. The motor windings and their associated resistances and inductances are needed to control the motor. The controlled motor will use a certain amount of power and will produce a certain torque. Thus, the design input is coupled to the control output and the control input is coupled to the design output. The results for the fully coupled system are different than for a partially coupled system, where the design input does not depend on the control output. Most of the literature uses the partially coupled system exclusively. In a related article, we demonstrate the equivalence of some solution strategies for partially coupled systems [9]. However, further improvement can be found for most systems when a fully coupled system model is used.

SOLUTION STRATEGIES

The combined optimization strategies are typical formulations for the solution of systems involving both an embodiment design problem and a control problem. These strategies are explained more fully in Reyer and Papalambros [6]. The traditional method of sequentially optimizing the design then optimizing the control is called the *Single Pass Strategy.* Since the single pass formulation requires one set of coupling quantities be fixed, an improvement is to modify those parameters via the *Iterative Strategy* — repetitively designing and controlling until the coupling quantities agree. The *Decoupled System Strategy* optimizes the entire design and control system while fixing one set of coupling quantities. The *Concurrent Strategy* or "All At Once" optimizes the system treating all coupling quantities as variables and attempting to solve Eq. (3) directly. The *Partition Strategy* treats the optimal design and optimal control as separate subproblems with a master problem to coordinate the system optimization. The *Bilevel Strategy* (or nested strategy) treats the optimal control as a subproblem and optimizes the system assuming that the optimal system will have optimal controller gains.

OPTIMALITY CONDITIONS

This section examines the optimality conditions of the decision models from the various solution strategies. The Karush-Kuhn-Tucker conditions provide necessary conditions for a point to be declared an optimum and form the basis for many numerical optimization techniques. Since optimality conditions are used to determine whether a point is an optimum, equivalent strategies should have the same optimality conditions. The conditions prove the equivalence of the Concurrent and Partition Strategies.

However, inconsistencies are shown with the remaining strategies. The inconsistencies are further examined to determine if the strategies are equivalent for special cases such as partially coupled or non-coupled problems.

Concurrent Strategy

The system perspective of the combined optimal embodiment design and control problem is the Concurrent Strategy. Using the decision model from the Concurrent Strategy, Eq. (3), system optimality conditions can be developed. The first step is to define the Lagrangian equation using Lagrange multipliers for each of the constraints. The multipliers are λ for equality constraints, with a superscript denoting with which constraints a specific λ is associated, and μ for inequality constraints with a similar superscript notation. For a more compact notation the functional dependence on the variables is not listed for each equation. See Eq. (3) for those relations. Also note that the proper vector notation is: $(\lambda^g)^T\boldsymbol{g}$; however, for a more compact notation the transposes are not shown and $(\lambda^g)^T\boldsymbol{g}$ is shown as $\lambda^g\boldsymbol{g}$. The superscripts on the Lagrange multipliers refer to the constraints with which they are associated. This should not be confused with $\partial\boldsymbol{g}/\partial\boldsymbol{b} = \boldsymbol{g}_b$, in which the subscript denotes a partial derivative.

Using this notation, the Lagrangian for the problem stated in Eq. (3) follows.

$$L(\boldsymbol{b},\boldsymbol{d},\boldsymbol{p},t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{a},\boldsymbol{u}) = \varpi_1 f + \mu^g\boldsymbol{g} + \lambda^h\boldsymbol{h} + \mu^\gamma\gamma + \int_{t_0}^{t_f}[\varpi_2 j + \mu^l\boldsymbol{l} + \lambda^k\boldsymbol{k} + \lambda^r(\boldsymbol{r}-\dot{\boldsymbol{x}}) + \mu^\beta\beta + \lambda^\gamma\zeta]dt \tag{4}$$

Note that the dynamic system output equation, Eq. (5), does not appear in this Lagrangian.

$$\boldsymbol{y} = \boldsymbol{s}(\boldsymbol{x}(t), \boldsymbol{z}(t), t, \boldsymbol{v}; \boldsymbol{u}) \tag{5}$$

The output equation does not determine any variables of interest and the output is not used in any other equations. Here it is assumed that the control signal can be represented in terms of the state variables ***x(t)***. More detailed control strategies would also include the output equation in modeling an observer.

The integration by parts technique is used to remove the dependence of the Lagrangian on $\dot{\boldsymbol{x}}(t)$. The Lagrangian may then be rewritten:

$$L = \varpi_1 f + \mu^g\boldsymbol{g} + \lambda^h\boldsymbol{h} + \mu^\gamma\gamma - \left[\lambda^r\boldsymbol{x}\right]\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f}[\varpi_2 j + \mu^l\boldsymbol{l} + \lambda^k\boldsymbol{k} + \lambda^r\boldsymbol{r} + \dot{\lambda}^r\boldsymbol{x} + \mu^\beta\beta + \lambda^\gamma\zeta]dt \tag{6}$$

A necessary condition for optimality is that the partial derivatives of the Lagrangian must vanish at the optimal point. The partial derivatives follow.

$$\frac{\partial L}{\partial \boldsymbol{b}} = \varpi_1 f_b + \mu^g\boldsymbol{g}_b + \lambda^h h_b + \mu^\gamma\gamma_b - \int_{t_0}^{t_f}\mu^\beta\beta_b dt \tag{7}$$

$$\frac{\partial L}{\partial \boldsymbol{d}} = \varpi_1 f_d + \mu^g\boldsymbol{g}_d + \lambda^\eta\boldsymbol{h}_d + \mu^\gamma\gamma_d \tag{8}$$

$$\frac{\partial L}{\partial \boldsymbol{p}} = \int_{t_0}^{t_f}\left[\lambda^\zeta\zeta_p\right]dt \tag{9}$$

$$\frac{\partial L}{\partial \boldsymbol{v}} = -\mu^\gamma\gamma_v + \int_{t_0}^{t_f}\left[\varpi_2 j_v + \mu^l\boldsymbol{l}_v + \lambda^k\boldsymbol{k}_v + \lambda^r\boldsymbol{r}_v + \mu^\beta\beta_v + \lambda^\zeta\zeta_v\right]dt \tag{10}$$

$$\frac{\partial L}{\partial \boldsymbol{x}} = -\left[\lambda^r\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right]\Bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f}\left[\varpi_2 j_x + \mu^l\boldsymbol{l}_x + \lambda^k\boldsymbol{k}_x + \lambda^r\boldsymbol{r}_x + \dot{\lambda}^r + \mu^\beta\beta_x - \lambda^\zeta\zeta_x\right]dt \tag{11}$$

$$\frac{\partial L}{\partial \boldsymbol{z}} = \int_{t_0}^{t_f}\left[\varpi_2 j_z + \mu^l\boldsymbol{l}_z + \lambda^k\boldsymbol{k}_z + \lambda^r\boldsymbol{r}_z + \mu^\beta\beta_z - \lambda^\zeta\zeta_z\right]dt \tag{12}$$

As internal variables, ***x(t)*** and ***z(t)*** could in theory be removed from the problem and would therefore not be part of the Lagrangian partial derivatives. In practice, the variables cannot be removed and the control literature typically includes their partial derivatives as part of the Pontryagin conditions, which are the dynamic equivalent of the KKT conditions. The KKT optimality conditions for the concurrent problem are stated in Table 1. Equation (15) in the optimality conditions states that the partial derivatives in Eqs. (7) – (12) vanish at the optimum.

Table 1: Concurrent Optimality Conditions

Feasibility	$\boldsymbol{g} \le \boldsymbol{0}$ $\boldsymbol{l} \le \boldsymbol{0}$ $\beta \le \boldsymbol{0}$ $\gamma \le \boldsymbol{0}$	$\boldsymbol{h} = \boldsymbol{0}$ $\boldsymbol{k} = \boldsymbol{0}$ $\boldsymbol{r} - \dot{\boldsymbol{x}} = \boldsymbol{0}$ $\zeta = \boldsymbol{0}$		(13)
Complementary Slackness	$\mu^g \boldsymbol{g} = \boldsymbol{0}$ $\mu^l \boldsymbol{l} = \boldsymbol{0}$ $\mu^\beta \beta = \boldsymbol{0}$ $\mu^\gamma \gamma = \boldsymbol{0}$	$\mu^g \ge \boldsymbol{0}$ $\mu^l(t) \ge \boldsymbol{0}$ $\mu^\beta(t) \ge \boldsymbol{0}$ $\mu^\gamma \ge \boldsymbol{0}$	$\lambda^h \ne \boldsymbol{0}$ $\lambda^k(t) \ne \boldsymbol{0}$ $\lambda^r(t) \ne \boldsymbol{0}$ $\lambda^\zeta(t) \ne \boldsymbol{0}$	(14)
Optimality	$dL = 0$ $\frac{\partial L}{\partial \boldsymbol{b}} = 0$ $\frac{\partial L}{\partial \boldsymbol{p}} = 0$ $\frac{\partial L}{\partial \boldsymbol{x}} = 0$	 $\frac{\partial L}{\partial \boldsymbol{d}} = 0$ $\frac{\partial L}{\partial \boldsymbol{v}} = 0$ $\frac{\partial L}{\partial \boldsymbol{z}} = 0$		(15)

These conditions are stated as is typical in the design literature; the controls literature has a few notational differences. In the controls literature the vanishing condition of Eq. (11) defines a ‘costate’ equation, Eq. (16).

$$\dot{\lambda}^r(t) = -\varpi_2 j_x - \mu^l \boldsymbol{l}_x - \lambda^k \boldsymbol{k}_x - \lambda^r \boldsymbol{r}_x - \mu^\beta \beta_x - \lambda^\zeta \zeta_x \qquad (16)$$

The costate equation is a first order linear differential equation for the Lagrange multiplier that is associated with the dynamics of the system.

Another common practice in the controls literature is to define a new quantity, the Hamiltonian, which permits simplification of the controls optimality conditions. The Hamiltonian is typically defined as $H = \varpi_2 j + \lambda^r \boldsymbol{r}$. In the concurrent problem, the full coupling and full complement of constraints limit the usefulness of the Hamiltonian notation for simplification purposes.

Finally, the typical problem in the controls literature is an optimal control problem, whereas the problem as stated here is an optimal gains problem. The optimal controls and optimal gains problems are related through the ζ constraint, which allows an assumed control strategy to define the controller $\boldsymbol{z}(t)$ as a function of the gains $\boldsymbol{p}$ and the states $\boldsymbol{x}(t)$. Except for the degenerate case, when the ζ equality constraint is inactive, the optimal gains problem does not yield an optimal controller. However, a true optimal control problem can be examined with slight modifications to the concurrent optimality conditions. Removal of the ζ constraint and elimination of the gains $\boldsymbol{p}$ deletes the multiplier λ^ζ term in Eqs. (10) – (12) and deletes Eq. (9) entirely. The remaining problem is the Concurrent Strategy for an optimal design problem combined with a “true” optimal control problem. If the optimal control problem is solved, then the next step is to determine a control strategy and find the gains that will approximately achieve the optimal controller. The formulation with the optimal gains problem eliminates the step in which the true optimal controller is found. The modifications presented here for reference purposes are based on control literature terminology. The paper will continue to follow the combined problem from the design point of view.

Partition Strategy

In Reyer [8], the decision model from the Concurrent Strategy is decomposed to form the Partition Strategy. The partitioned decision model has a master problem (MP), a design subproblem (DSP), and a control subproblem (CSP) as shown in Figure 1. The master problem varies the linking variables $\boldsymbol{b}$ and $\boldsymbol{v}$ to minimize the system objective with optimal subproblems. The design subproblem treats the linking variables as parameters and solves an optimal design problem for the design variables $\boldsymbol{d}^*$. The control subproblem treats the linking variables as parameters and solves an optimal gain problem for the controller gains $\boldsymbol{p}^*$, the dynamics $\boldsymbol{x}(t)$, and the controller signal $\boldsymbol{z}(t)$. A method for coordinating the decomposed model also appears in Reyer [8]. This section examines the optimality conditions for the Partition Strategy.

The first step in determining the optimality conditions is to define the system Lagrangian. The multipliers are defined similarly to the concurrent problem. The multipliers are: λ for equality constraints, with a superscript denoting with which

Master Problem (MP)

$$\min_{\boldsymbol{b},\boldsymbol{v}} \quad \varpi_1 f(\boldsymbol{b};\boldsymbol{a},\boldsymbol{d}^*) + \varpi_2 J\left(\boldsymbol{v};t,\boldsymbol{u},\boldsymbol{x}^*(t),\boldsymbol{z}^*(t)\right)$$

Design Subproblem (DSP)

$$\min_{\boldsymbol{d}} \quad \varpi_1 f(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b})$$

subject to:

$$\boldsymbol{g}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) \le \boldsymbol{0}$$
$$\boldsymbol{h}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{0}$$
$$\gamma: \quad \boldsymbol{V}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) - \boldsymbol{v} \le \boldsymbol{0}$$

Control Subproblem (CSP)

$$\min_{\boldsymbol{p},t,\boldsymbol{x}(t),\boldsymbol{z}(t)} \quad \varpi_2 J\left(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}\right)$$

subject to:

$$\boldsymbol{k}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{0}$$
$$\boldsymbol{l}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) \le \boldsymbol{0}$$
$$\boldsymbol{r}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \dot{\boldsymbol{x}} = \boldsymbol{0}$$
$$s(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - y = \boldsymbol{0}$$
$$\zeta: \quad \boldsymbol{Z}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{z}(t) = \boldsymbol{0}$$
$$\beta: \quad \boldsymbol{B}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{b} \le \boldsymbol{0}$$

Figure 1. Partition Strategy Formulation

constraints a specific λ is associated, and μ for inequality constraints with a similar superscript notation. Each part, the two subproblems and the master problem, has its own Lagrangian as defined below.

$$L_{MP}(\boldsymbol{b},\boldsymbol{v};\boldsymbol{a},\boldsymbol{d},\boldsymbol{p}^*,t^*,\boldsymbol{u},\boldsymbol{x}^*(t),\boldsymbol{z}^*(t)) = \varpi_1 f + \int_{t_0}^{t_f}\left[\varpi_2 j\right]dt \tag{17}$$

$$L_{DSP}(\boldsymbol{d};\boldsymbol{b},\boldsymbol{v}) = \varpi_1 f + \mu^g \boldsymbol{g} + \lambda^h \boldsymbol{h} + \mu^\gamma \gamma \tag{18}$$

$$\begin{aligned} L_{CSP}(\boldsymbol{p},t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{b},\boldsymbol{u},\boldsymbol{v}) &= -\left[\lambda^r \boldsymbol{x}\right]\Big|_{t_0}^{t_f} \\ &+\int_{t_0}^{t_f}[\varpi_2 j + \mu^l l + \lambda^k k + \lambda^r \boldsymbol{r} + \dot{\lambda}^r \boldsymbol{x} + \mu^\beta \beta + \lambda^\gamma \zeta]dt \end{aligned} \tag{19}$$

The system Lagrangian is the sum of the Lagrangians of the parts, Eq. (20).

$$L = L_{MP} + L_{DSP} + L_{CSP} \tag{20}$$

Once again the partial derivatives of the Lagrangian must be found and set equal to zero for a solution to be optimal. The variable $\boldsymbol{d}$ only affects the design subproblem.

$$\frac{\partial L}{\partial \boldsymbol{d}} = \cancelto{0}{\frac{\partial L_{MP}}{\partial \boldsymbol{d}}} + \frac{\partial L_{DSP}}{\partial \boldsymbol{d}} + \cancelto{0}{\frac{\partial L_{CSP}}{\partial \boldsymbol{d}}} = \varpi_1 f_d + \mu^g \boldsymbol{g}_d + \lambda^h \boldsymbol{h}_d + \mu^\gamma \gamma_d \tag{21}$$

The variables $\boldsymbol{p}$, $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ are determined in the control subproblem.

$$\frac{\partial L}{\partial \boldsymbol{p}} = \cancelto{0}{\frac{\partial L_{MP}}{\partial \boldsymbol{p}}} + \cancelto{0}{\frac{\partial L_{DSP}}{\partial \boldsymbol{p}}} + \frac{\partial L_{CSP}}{\partial \boldsymbol{p}} = \int_{t_0}^{t_f}\left[\lambda^\zeta \zeta_p\right]dt \tag{22}$$

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{x}} &= \cancelto{0}{\frac{\partial L_{MP}}{\partial \boldsymbol{x}}} + \cancelto{0}{\frac{\partial L_{DSP}}{\partial \boldsymbol{x}}} + \frac{\partial L_{CSP}}{\partial \boldsymbol{x}} \\ &= -\left[\lambda^r \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right]\Bigg|_{t_0}^{t_f} \\ &+\int_{t_0}^{t_f}\left[\varpi_2 j_x + \mu^l \boldsymbol{l}_x + \lambda^k \boldsymbol{k}_x + \lambda^r \boldsymbol{r}_x + \dot{\lambda}^r + \mu^\beta \beta_x - \lambda^\zeta \zeta_x\right]dt \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{z}} &= \cancelto{0}{\frac{\partial L_{MP}}{\partial \boldsymbol{z}}} + \cancelto{0}{\frac{\partial L_{DSP}}{\partial \boldsymbol{z}}} + \frac{\partial L_{CSP}}{\partial \boldsymbol{z}} \\ &= \int_{t_0}^{t_f}\left[\varpi_2 j_z + \mu^l \boldsymbol{l}_z + \lambda^k \boldsymbol{k}_z + \lambda^r \boldsymbol{r}_z + \mu^\beta \beta_z - \lambda^\zeta \zeta_z\right]dt \end{aligned} \tag{24}$$

The linking variables are determined in the master problem. However, they also appear as parameters in the subproblems, making the optimal local variables of the subproblems dependent on the linking variables. So the partial derivative of the objective in the master problem with respect to $\boldsymbol{b}$ includes the variations of the subproblem objectives with respect to the local variables multiplied by the variation of the local variables with respect to the linking variables $\boldsymbol{b}$.

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{b}} &= \frac{\partial L_{MP}}{\partial \boldsymbol{b}} + \cancelto{0}{\frac{\partial L_{DSP}}{\partial \boldsymbol{b}}} + \cancelto{0}{\frac{\partial L_{CSP}}{\partial \boldsymbol{b}}} \\ &= \varpi_1 f_b + \varpi_2 J_b + \varpi_1 f_d \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{b}} + \varpi_2 J_p \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{b}} + \varpi_2 J_x \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{b}} + \varpi_2 J_z \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{b}} \end{aligned} \tag{25}$$

To develop the $\boldsymbol{b}$ partial derivative fully, consider the $\boldsymbol{d}$ variable term. Since the design subproblem is optimized, the derivative of its Lagrangian with respect to the design variables must be zero. The equation is solved for $\varpi_1 f_d$.

$$\frac{\partial L_{DSP}}{\partial \boldsymbol{d}} = 0 \qquad \varpi_1 f_d = -\mu^g \boldsymbol{g}_d - \lambda^h \boldsymbol{h}_d - \mu^\gamma \gamma_d \tag{26}$$

In the complementary slackness condition, the Lagrange multipliers times the constraints must be zero. Differentiating the complementary slackness condition with respect to $\boldsymbol{b}$ gives a new relation.

$$\mu^g \boldsymbol{g} = 0 \qquad \mu^g \boldsymbol{g}_b + \mu^g \boldsymbol{g}_d \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{b}} = 0 \qquad \mu^g \boldsymbol{g}_b = -\mu^g \boldsymbol{g}_d \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{b}} \tag{27}$$

Combining Eqs. (26) and (27) yields a term for Eq. (25).

$$\varpi_1 f_d \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{b}} = \mu^g \boldsymbol{g}_b + \lambda^h \boldsymbol{h}_b + \mu^\gamma \gamma_b \tag{28}$$

A similar development is followed to determine the result of the control subproblem variations.

$$\varpi_2 J_p \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{b}} + \varpi_2 J_x \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{b}} + \varpi_2 J_z \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{b}} = -\int_{t_0}^{t_f} \mu^\beta \beta_b dt \tag{29}$$

Equations (25), (28) and (29) combine to form the partial derivative of the system Lagrangian with respect to the linking variable $\boldsymbol{b}$.

$$\frac{\partial L}{\partial \boldsymbol{b}} = \varpi_1 f_b + \mu^g \boldsymbol{g}_b + \lambda^h h_b + \mu^\gamma \gamma_b - \int_{t_0}^{t_f} \mu^\beta \beta_b dt \tag{30}$$

The partial derivative of the Lagrangian with respect to $\boldsymbol{v}$ follows a development that is similar to the partial derivative with respect to $\boldsymbol{b}$. The partial derivative of the Lagrangian reflects the dependence of the local variables on the linking variables, Eq. (31).

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{v}} &= \frac{\partial L_{MP}}{\partial \boldsymbol{v}} + \cancelto{0}{\frac{\partial L_{DSP}}{\partial \boldsymbol{v}}} + \cancelto{0}{\frac{\partial L_{CSP}}{\partial \boldsymbol{v}}} \\ &= \varpi_1 f_v + \varpi_2 J_v + \varpi_1 f_d \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{v}} + \varpi_2 J_p \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} + \varpi_2 J_x \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{v}} + \varpi_2 J_z \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{v}} \end{aligned} \tag{31}$$

The subproblem variation terms are found.

$$\varpi_1 f_v \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{v}} = -\mu^\gamma \gamma_v \tag{32}$$

Table 2: Partition Strategy Optimality Conditions

Feasibility	$\left.\begin{array}{l} \boldsymbol{g} \le \boldsymbol{0} \\ \boldsymbol{h} = \boldsymbol{0} \\ \gamma \le \boldsymbol{0} \end{array}\right\} DSP$	$\left.\begin{array}{r} \boldsymbol{k} = \boldsymbol{0} \\ \boldsymbol{l} \le \boldsymbol{0} \\ \boldsymbol{r} - \dot{\boldsymbol{x}} = \boldsymbol{0} \\ \zeta = \boldsymbol{0} \\ \beta \le \boldsymbol{0} \end{array}\right\} CSP$
Complementary Slackness	$\left.\begin{array}{ll} \mu^g \boldsymbol{g} = \boldsymbol{0} & \mu^g \ge \boldsymbol{0} \\ \mu^\gamma \gamma = \boldsymbol{0} & \mu^\gamma \ge \boldsymbol{0} \\ & \lambda^h \ne \boldsymbol{0} \end{array}\right\} \mathrm{DSP}$ $\left.\begin{array}{ll} \mu^l \boldsymbol{l} = \boldsymbol{0} & \mu^l(t) \ge \boldsymbol{0} \\ \mu^\beta \beta = \boldsymbol{0} & \mu^\beta(t) \ge \boldsymbol{0} \\ & \lambda^k(t) \ne \boldsymbol{0} \\ & \lambda^r(t) \ne \boldsymbol{0} \\ & \lambda^\zeta(t) \ne \boldsymbol{0} \end{array}\right\} CSP$	
Optimality	$dL = 0$ $\frac{\partial L}{\partial \boldsymbol{b}} = 0 \quad \frac{\partial L}{\partial \boldsymbol{d}} = 0$ $\frac{\partial L}{\partial \boldsymbol{p}} = 0 \quad \frac{\partial L}{\partial \boldsymbol{v}} = 0$ $\frac{\partial L}{\partial \boldsymbol{x}} = 0 \quad \frac{\partial L}{\partial \boldsymbol{z}} = 0$	

$$\varpi_2 J_p \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} + \varpi_2 J_x \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{v}} + \varpi_2 J_z \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{v}} = \int_{t_0}^{t_f} \left[\mu^l \boldsymbol{l}_v + \lambda^k \boldsymbol{k}_v + \lambda^r \boldsymbol{r}_v + \mu^\beta \beta_v + \lambda^\zeta \zeta_v \right] dt \tag{33}$$

Combining the equations yields the partial derivative of the system Lagrangian with respect to $\boldsymbol{v}$.

$$\frac{\partial L}{\partial \boldsymbol{v}} = -\mu^\gamma \gamma_v + \int_{t_0}^{t_f} \left[\varpi_2 j_v + \mu^l \boldsymbol{l}_v + \lambda^k \boldsymbol{k}_v + \lambda^r \boldsymbol{r}_v + \mu^\beta \beta_v + \lambda^\zeta \zeta_v \right] dt \tag{34}$$

The optimality conditions for the partition strategy include the feasibility and complementary slackness equations from both the design and control subproblems, the costate equations from the control subproblems and the vanishing condition of the partial derivatives of the Lagrangian. The conditions are summarized in Table 2. Obviously the conditions for the Partition Strategy are identical to those from the Concurrent Strategy and the strategies are equivalent in terms of the solution set defined by these conditions.

Other Combined Strategies

The optimality conditions of the remaining combined strategies, the Decoupled System and Bilevel Strategies, are now examined. The decision model for the Decoupled system strategy appears in Eq. (35).

$$\begin{array}{ll} \min\limits_{\boldsymbol{d},\,\boldsymbol{p},\,t,\,\boldsymbol{v},\,\boldsymbol{x}(t),\,\boldsymbol{z}(t)} & \varpi_1 f(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) + \varpi_2 \int_{t_0}^{t_f} j(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u})\,dt \\ \text{mubject to:} & \\ & \boldsymbol{g}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) \le \boldsymbol{0} \\ & \boldsymbol{h}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{0} \\ & \boldsymbol{k}(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u}) = \boldsymbol{0} \\ & \boldsymbol{l}(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u}) \le v \\ & \boldsymbol{r}(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u}) - \dot{\boldsymbol{x}} = \boldsymbol{0} \\ & s(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u}) - y = \boldsymbol{0} \\ \beta: & \boldsymbol{B}(t,\boldsymbol{v},\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u}) - \boldsymbol{b} \le \boldsymbol{0} \\ \gamma: & \boldsymbol{V}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) - \boldsymbol{v} = \boldsymbol{0} \\ \zeta: & \boldsymbol{Z}(\boldsymbol{p},t,\boldsymbol{v},\boldsymbol{x}(t);\boldsymbol{u}) - \boldsymbol{z}(t) = \boldsymbol{0} \end{array} \tag{35}$$

The decision model for the Bilevel Strategy appears in Eq. (36).

$$\begin{array}{ll} \min\limits_{\boldsymbol{b},\boldsymbol{d},\boldsymbol{v}} & \varpi_1 f(\boldsymbol{b},\boldsymbol{d};\boldsymbol{a}) + \varpi_2 \int_{t_0}^{t_f} j\left(\boldsymbol{v};t^*,\boldsymbol{u},\boldsymbol{x}^*(t),\boldsymbol{z}^*(t)\right) dt \\ \text{subject to:} & \\ & \boldsymbol{g}(\boldsymbol{b},\boldsymbol{d};\boldsymbol{a}) \le \boldsymbol{0} \\ & \boldsymbol{h}(\boldsymbol{b},\boldsymbol{d};\boldsymbol{a}) = \boldsymbol{0} \\ \beta: & \boldsymbol{B}\left(\boldsymbol{v};t^*,\boldsymbol{u},\boldsymbol{x}^*(t),\boldsymbol{z}^*(t)\right) - \boldsymbol{b} \le \boldsymbol{0} \\ \gamma: & \boldsymbol{V}(\boldsymbol{b},\boldsymbol{d};\boldsymbol{a}) - \boldsymbol{v} \le \boldsymbol{0} \\ & \min\limits_{\boldsymbol{p},t,\boldsymbol{x}(t),\boldsymbol{z}(t)} \ \varpi_2 \int_{t_0}^{t_f} j(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v})\,dt \\ & \text{subject to} \\ & \quad \boldsymbol{k}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{0} \\ & \quad \boldsymbol{l}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) \le \boldsymbol{0} \\ & \quad \boldsymbol{r}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \dot{\boldsymbol{x}}(t) = \boldsymbol{0} \\ & \quad s(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - y(t) = \boldsymbol{0} \\ \zeta: & \quad \boldsymbol{Z}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{z}(t) = \boldsymbol{0} \end{array} \tag{36}$$

The constraints are similar to those in the Concurrent and Partition Strategies; the single exception is the γ constraint, which is an equality constraint in the Decoupled System Strategy instead of an inequality. The associated feasibility and complementary slackness conditions then must agree with those from the Concurrent Strategy, except that in the Decoupled System Strategy the μ^g term is an λ^g term. The main difference between these strategies and the Concurrent Strategy is the organization of the equations, which affects their Lagrangian and its partial derivatives.

For the Decoupled System Strategy, the partial derivatives follow.

$$\frac{\partial L}{\partial \boldsymbol{d}} = \varpi_1 f_d + \mu^g \boldsymbol{g}_d + \lambda^\eta \boldsymbol{h}_d + \lambda^\gamma \gamma_d \tag{37}$$

$$\frac{\partial L}{\partial \boldsymbol{p}} = \int_{t_0}^{t_f} \left[\lambda^\zeta \zeta_p\right] dt \tag{38}$$

$$\frac{\partial L}{\partial \boldsymbol{v}} = -\lambda^\gamma \gamma_v + \int_{t_0}^{t_f} \left[\varpi_2 j_v + \mu^l \boldsymbol{l}_v + \lambda^k \boldsymbol{k}_v + \lambda^r \boldsymbol{r}_v + \mu^\beta \beta_v + \lambda^\zeta \zeta_v\right] dt \tag{39}$$

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{x}} = &-\left[\lambda^r \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right]\Bigg|_{t_0}^{t_f} \\ &+ \int_{t_0}^{t_f} \left[\varpi_2 j_x + \mu^l \boldsymbol{l}_x + \lambda^k \boldsymbol{k}_x + \lambda^r \boldsymbol{r}_x + \dot{\lambda}^r + \mu^\beta \beta_x - \lambda^\zeta \zeta_x\right] dt \end{aligned} \tag{40}$$

$$\frac{\partial L}{\partial \boldsymbol{z}} = \int_{t_0}^{t_f} \left[\varpi_2 j_z + \mu^l \boldsymbol{l}_z + \lambda^k \boldsymbol{k}_z + \lambda^r \boldsymbol{r}_z + \mu^\beta \beta_z + \lambda^\zeta \zeta_z\right] dt \tag{41}$$

Notice that these equations are identical to Eqs. (8) – (12) from the concurrent problem derivation. The difference with the concurrent problem is the omission of $\boldsymbol{b}$ as a variable and therefore the omission of its Lagrangian partial derivative equation. The effect of this omission is that the decoupled system can reach the concurrent solution only if $\boldsymbol{b}$ is perfectly chosen and all of the γ constraints are active.

The Bilevel Strategy partial derivatives appear below.

$$\frac{\partial L_{MP}}{\partial \boldsymbol{b}} = \varpi_1 f_b + \mu^g \boldsymbol{g}_b + \lambda^h h_b + \mu^\gamma \gamma_b + \int_{t_0}^{t_f} \mu^\beta \beta_b dt \tag{42}$$

$$\frac{\partial L_{MP}}{\partial \boldsymbol{d}} = \varpi_1 f_d + \mu^g \boldsymbol{g}_d + \lambda^\eta \boldsymbol{h}_d + \mu^\gamma \gamma_d \tag{43}$$

$$\frac{\partial L_{SP}}{\partial \boldsymbol{p}} = \int_{t_0}^{t_f} \left[\lambda^\zeta \zeta_p\right] dt \tag{44}$$

$$\begin{aligned} \frac{\partial L_{MP}}{\partial \boldsymbol{v}} = &-\mu^\gamma \gamma_v + \int_{t_0}^{t_f} \left[\varpi_2 j_v + \mu^l \boldsymbol{l}_v + \lambda^k \boldsymbol{k}_v + \lambda^r \boldsymbol{r}_v + \lambda^\zeta \zeta_v\right] dt \\ &+ \int_{t_0}^{t_f} \left[\mu^\beta \left(\beta_v + \beta_x \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{v}} + \beta_z \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{v}} + \beta_p \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}\right)\right] dt \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{\partial L_{SP}}{\partial \boldsymbol{x}} = &-\left[\lambda^r \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right]\Bigg|_{t_0}^{t_f} \\ &+ \int_{t_0}^{t_f} \left[\varpi_2 j_x + \mu^l \boldsymbol{l}_x + \lambda^k \boldsymbol{k}_x + \lambda^r \boldsymbol{r}_x + \dot{\lambda}^r - \lambda^\zeta \zeta_x\right] dt \end{aligned} \tag{46}$$

$$\frac{\partial L_{SP}}{\partial \boldsymbol{z}} = \int_{t_0}^{t_f} \left[\varpi_2 j_z + \mu^l \boldsymbol{l}_z + \lambda^k \boldsymbol{k}_z + \lambda^r \boldsymbol{r}_z - \lambda^\zeta \zeta_z\right] dt \tag{47}$$

The Bilevel partial derivative equations have some common terms with the Concurrent Strategy. The two strategies match in the partial derivatives with respect to variables $\boldsymbol{d}$ and $\boldsymbol{b}$. In the optimal gain subproblem equations most of the terms match with one notable exception. Since the β coupling constraints are part of the master problem, the Lagrangian of the subproblem and its partial derivatives do not possess those terms, allowing subproblem solutions that are infeasible in the master problem. Furthermore, since the terms in the β constraint are functions of the variables at the subproblem optimum, the partial derivative with respect to variable $\boldsymbol{v}$ must include extra partial derivatives of the subproblem terms that are dependent on the linking variable $\boldsymbol{v}$. The only ways that the Bilevel Strategy solution can match the concurrent solution are if the β coupling constraints are either inactive or do not exist. Alternatively, the bilevel strategy could be changed by moving the β coupling constraints into the optimal gain subproblem: making another valid partition strategy. Moving the constraints, however, would make the method less attractive, since its purpose is usually to permit the direct analytical solution of an optimal gain problem. The Bilevel Strategy will only succeed if there are no β constraints, if the β constraints are inactive, or if those constraints are repositioned in the decision model.

The Decoupled System and Bilevel Strategies generally do not find the concurrent solution, though special cases of problems can be found for which the strategies achieve the correct solution.

Sequential Strategies

The sequential strategies, the Single Pass and Iterative Strategies, obviously do not have the same optimality conditions as the Concurrent Strategy, since there are no system level problems. Without a system level, the system Lagrangian cannot be defined. Both the Single Pass and Iterative Strategies treat the design and control as separate problems, Figure 2.

Design Problem	Control Problem
$\min_{\boldsymbol{d}} \quad \varpi_1 f(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b})$	$\min_{\boldsymbol{p},t,\boldsymbol{x}(t),\boldsymbol{z}(t)} \quad \varpi_2 \int_{t_0}^{t_f} \left(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}\right) dt$
subject to:	subject to:
$\boldsymbol{g}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) \le \boldsymbol{0}$	$\boldsymbol{k}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{0}$
$\boldsymbol{h}(\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{0}$	$\boldsymbol{l}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) \le \boldsymbol{0}$
	$\boldsymbol{r}(t,\boldsymbol{x}(t),\boldsymbol{z}(t);\boldsymbol{u},\boldsymbol{v}) - \dot{\boldsymbol{x}} = \boldsymbol{0}$
	$\zeta: \quad \boldsymbol{Z}(\boldsymbol{p},t,\boldsymbol{x}(t);\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{z}(t) = \boldsymbol{0}$

Figure 2. Sequential Strategies' Model

The optimality conditions for those problems are in the following equations.

Design Problem

Table 3: Conditions for Finding the True Optima

If your system is:		Fully Coupled				Partially Coupled (no $\boldsymbol{b}$, β)	
And you have additional information:		no more info	All β inactive	β constraints solved within optimal gain subproblem	Perfect $\boldsymbol{b}$	no more info	all γ active
Solution Strategies	All At Once	√	√	√	N/A	√	√
	Partition	√	√	√	N/A	√	√
	Decoupled System				√	√	√
	Bilevel		√	√	N/A	√	√
	Iterative				N/A		√

Note: √ indicates that the optimality conditions for a solution strategy match those for the Concurrent Strategy with the listed requirements. The statements in this table are not intended to indicate ease or difficultly in solving the problem.

$$\frac{\partial L_{DP}}{\partial \boldsymbol{d}} = \varpi_1 f_d + \mu^g \boldsymbol{g}_d + \lambda^h \boldsymbol{h}_d \tag{48}$$

Control Problem

$$\frac{\partial L_{CP}}{\partial \boldsymbol{p}} = \int_{t_0}^{t_f} \left[\lambda^\zeta \zeta_p\right] dt \tag{49}$$

$$\frac{\partial L_{CP}}{\partial \boldsymbol{x}} = -\left[\lambda^r \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right]\Bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[\varpi_2 j_x + \mu^l \boldsymbol{l}_x + \lambda^k \boldsymbol{k}_x + \lambda^r \boldsymbol{r}_x + \dot{\lambda}^r - \lambda^\zeta \zeta_x\right] dt \tag{50}$$

$$\frac{\partial L_{CP}}{\partial \boldsymbol{z}} = \int_{t_0}^{t_f} \left[\varpi_2 j_z + \mu^l \boldsymbol{l}_z + \lambda^k \boldsymbol{k}_z + \lambda^r \boldsymbol{r}_z - \lambda^\zeta \zeta_z\right] dt \tag{51}$$

The remaining two sequential strategies differ slightly. The Single Pass performs exactly one optimization of the design problem and the control problem and checks that the β constraint is satisfied. The Iterative Strategy repeatedly optimizes the two parts until the coupling constraints are satisfied as equalities. One could consider the sequential problems as having two subproblems and a master problem. For the Iterative Strategy the master problem could be formulated in several ways: e.g., a problem with a null objective and two constraints or a problem with a sum of squares of the constraints. In either case the strategy still does not approach the Concurrent solution. The tradeoffs between improving the design and improving the control simply cannot be delineated.

SUMMARY

We have compared six strategies that are used to solve a general combined optimal embodiment design and control problem. The problem is fully coupled in that the results and inputs to the design and control problems depend on each other. Since optimality conditions are used to declare whether a point is optimal for a problem, if the strategies are equivalent, the optimality conditions should be equivalent. Thus, comparison of optimality conditions indicates whether a strategy will find system optima.

The concurrent and Partition Strategies find the true optima for the general problem. Yet the other strategies have been used successfully to solve various examples. Further examination of the optimality conditions allows us to explore special cases under which other strategies do find the true optima. Table 3 summarizes the conditions under which the true optima are found.

These results have some limitations. Having the correct optimality conditions does not provide any indication as to whether the problem could actually be solved. In fact, we have found that the Concurrent Strategy can be very difficult to solve and the Partition Strategy is solvable only when you can determine an appropriate coordination strategy. From these comparisons we do not know how far from the true optima the unsuccessful strategies' results will be. Practically speaking, even the unsuccessful strategies will give a better answer than not considering system optimization at all. Whether the results are ‘good enough’ is certainly problem dependent.

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