OPTIMAL STRUCTURAL TOPOLOGY DESIGN USING THE HOMOGENIZATION METHOD WITH MULTIPLE CONSTRAINTS

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In applications of the homogenization method for optimal structural topology design the solution is obtained by solving the optimality conditions directly. This reduces the computational burden by taking advantage of closed-form solutions but it restricts the optimization model to having only one constraint. The article develops a generalized class of convex approximation methods for mathematical programming that can be used for the optimal topology homogenization problem with multiple constraints included in the model, without substantial reduction in computational efficiency. A richer class of design models can be then addressed using the homogenization method. Design examples illustrate the performance of the proposed solution strategy.

Keywords: Convex approximation method; homogenization; structural topology optimization; multipurpose topology design.

INTRODUCTION

Structural topology optimization has received substantial attention in recent years and several successful approaches have been reported (e.g. Bendsøe and Mota Soares [1], Gilmore et al. [2]). A now well-developed method initially proposed by Bendsøe and Kikuchi [3] uses a homogenization formulation in the structural model. This results in optimization models with a large number of design variables, typically 5,000 or more, associated with the finite elements used for the structural analysis needed to compute model functions. General purpose
mathematical programming algorithms have prohibitive computational costs for such models. The attractive alternative is the so-called optimality criteria method, namely, to solve the optimality conditions directly provided some closed-form expressions can be derived. In the topology model this can be obtained if a single constraint is included in the optimization model (in addition to the equilibrium equations). This restriction limits the utility of the homogenization formulation.

Convex approximation methods have been widely used in structural size and shape optimization; they are well tailored for these models and provide fast computation [4–8]. For recent reviews on function approximations in structural optimization applications see Barthelemy and Haftka [9,10]. The similarity between optimality criteria and convex approximation methods had been noticed early [11]. As will be shown in the sequel, a generalization of previous convex approximation formulations has a strong similarity with results obtained via the optimality criteria formulation in topology problems as well. Therefore, generalized convex approximation methods can be used to solve topology optimization models using homogenization. An important result of this solution strategy is that multiple constraints can be now included directly in the topology optimization model and homogenization models can be used for a substantially larger class of design problems.

TOPOLOGY OPTIMIZATION USING HOMOGENIZATION

The main idea in topology optimization with homogenization is to compute an optimal material distribution on a given design domain, under given loads and boundary conditions. The distributed material is constructed by periodic microstructures. A typical microstructure model [12] consists of square unit cells with rectangular holes, Figure 1. Sizes \(a\) and \(b\) as well as the orientation angle \(\theta\) of each hole are treated as design variables. Here \(a = b = 1\) represents solid material, and \(a = b = 0\) represents a void. The average elastic tensor \(E_{ijkl}^G\) of the anisotropic material determined by homogenization is a function of \(a\), \(b\) and \(\theta\).
The topology optimization model is defined as

**minimize**: mean compliance  
**subject to**: equilibrium equations  
volume constraint  
and  
bound constraints on the design variables

In the two dimensional case the model is

$$
\text{minimize } f_0(a, b, \theta) = \sum_{i=1}^{2} \int_{\Omega} f_i u_i d\Omega + \sum_{i=1}^{2} \int_{\Gamma_r} t_i u_i d\Gamma
$$

subject to  
equilibrium equations

$$
f_i(a, b) = \int_{\Omega} (a + b - ab) d\Omega - \Omega_s \leq 0
$$

$$
0 \leq a \leq 1, \quad 0 \leq b \leq 1 \quad \text{and} \quad -\pi/2 \leq \theta \leq \pi/2
$$

(1)

where $a, b$ and $\theta$ are design variables, $f$ and $t$ are applied body forces in design domain $\Omega$ and tractions on a portion of specified boundary
\( \Gamma_T \) of the design domain, respectively, and \( \mathbf{u} \) is the vector of displacements.

Since \( \mathbf{u} \) is the solution of the equilibrium equations, assuming these are satisfied by solving the finite element method equations, the model is equivalent to

\[
\begin{align*}
\text{minimize} \quad & f_0(\mathbf{a}, \mathbf{b}, \theta) = \frac{1}{2} \sum_{(i,j,k,l=1)}^2 \int_{\Omega} E_{ijkl} \frac{\partial u_k}{\partial z_i} \frac{\partial u_l}{\partial z_j} d\Omega \\
\text{subject to} \quad & f_1(\mathbf{a}, \mathbf{b}) = \int_{\Omega} (a + b - ab) d\Omega - \Omega_s \leq 0 \\
\text{and} \quad & 0 \leq a \leq 1, \quad 0 \leq b \leq 1 \quad \text{and} \quad -\pi/2 \leq \theta \leq \pi/2
\end{align*}
\]

where \( \mathbf{z} \) is the vector of the position coordinates. This optimization model has only one constraint other than simple bound constraints.

An iterative scheme to solve this model and to update the design variables \( \mathbf{a} \) can be derived by optimality criteria [12]

\[
\mathbf{a}^{k+1} = \begin{cases} 
(\max \{1 - \zeta a_k, 0\} & \text{if } a^k(D_a^k)\eta \leq \max \{(1 - \zeta)a^k, 0\}) \\
(a^k(D_a^k)\eta \text{ if } \max\{(1 - \zeta)a^k, 0\} \leq a^k(D_a^k)\eta \leq \min\{(1 + \zeta)a^k, 1\}) \\
(\min\{(1 + \zeta)a^k, 1\} & \text{if } \min\{(1 + \zeta)a^k, 1\} \leq a^k(D_a^k)\eta 
\end{cases}
\]

(3)

and

\[
\mathbf{D}_a^k = \{\lambda^k(1 - \mathbf{b}^k)\}^{-1} \sum_{(i,j,k,l=1)}^2 \int_{\Omega} E_{ijkl} \frac{\partial u_k}{\partial a} \frac{\partial u_l}{\partial z_i} \frac{\partial u_l}{\partial z_j} d\Omega
\]

(4)

where \( \zeta \) is a move limit, \( \eta \) is a weighing factor and \( \lambda \) is the Langrange multiplier. The vector \( \mathbf{b} \) of design variables is updated in exactly the same way as \( \mathbf{a} \). Since the single constraint is active, the Langrange multiplier is updated in an inner loop using a bisection method to satisfy the volume constraint \( f_1(\mathbf{a}, \mathbf{b}) = 0 \).

Pederson [13] proved that the optimal \( \theta \) is identified with the directions of principal stresses. So \( \theta \) is adjusted to the directions after the design variables \( \mathbf{a} \) and \( \mathbf{b} \) are updated in each iteration.
CONVEX APPROXIMATION METHODS

Solving a structural optimization problem with a convex approximation method involves the following steps:

1. Give an initial design of the structure.
2. Analyze the structure to evaluate objective and constraint functions and their derivatives. Usually this means solving the equilibrium equations by the finite element method.
3. Linearize the primal objective and constraint functions with respect to some intermediate variables to obtain an approximate problem.
4. Derive the dual problem for the primal approximate one. Solve the dual problem to update the primal variables.
5. Go to step 2 until a given stopping criterion is satisfied.

Convex approximation methods have two main advantages: (a) structural analyses are not needed in solving the approximate problem, and (b) since the number of dual variables is equal to the number of constraints in the primal, the approximate problem can be solved efficiently when the number of constraints is much smaller than the number of design variables. This is often the case in optimal topology models.

Different choices of intermediate variables generate different convex approximations. The CONvex LINearization (CONLIN) method [4] uses direct or reciprocal variables as intermediate variables. If the first derivative of a function is positive (negative), the function is linearized with respect to the primal design variable $x_i$ (reciprocal of the primal variable $1/x_i$). The Method of Moving Asymptotes (MMA) [5] uses intermediate variables $1/(x_i - L_i)$ and $1/(U_i - x_i)$ for those with negative and positive derivatives, respectively. Lower and upper “moving asymptotes” $L_i$ and $U_i$ are used to adjust the curvature of the approximate functions and improve the accuracy of the approximation.

A more general class of convex approximation methods can be derived using $1/(U_{ji} - x_j)^{s_{ji}}$ and $1/(x_i - L_{ji})^{s_{ji}}$ as intermediate variables in Eq. (6) below. The weighting factors $s_{ji}$ can be adjusted to make the approximation more accurate. This new class can be developed as follows.

Consider the general optimization model
minimize $f_0(x)$  
subject to $f_j(x) \leq 0$ \hspace{1cm} j = 1, 2, \ldots, m$  
and $LB_i \leq x_i \leq UB_i$ \hspace{1cm} i = 1, 2, \ldots, n$ (5)

where $x = (x_1, \ldots, x_n)^T$ is a vector of design variables, $f_0(x)$ is the objective function, $f_j(x)$ are constraint functions, and $LB_i$ and $UB_i$ are given lower and upper bounds on the design variables. Using $1/(U_{ji} - x_i)^{s_{ji}}$ and $1/(x_i - L_{ji})^{s_{ji}}$ as intermediate variables, the approximate functions at point $x^k$ (at the $k$th iteration) are defined as

$$f^k_j(x) = r^k_j + \sum_{i=1}^{n} [P^k_{ji}(U^k_{ji} - x_i)^{s^k_{ji}} + Q^k_{ji}(x_i - L^k_{ji})^{s^k_{ji}}]$$ (6)

where

$$P^k_{ji} = \begin{cases} (1/s^k_{ji})(U^k_{ji} - x_i)^{s^k_{ji} + 1} \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}$$

$$Q^k_{ji} = \begin{cases} 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ -(1/s^k_{ji})(x_i - L^k_{ji})^{s^k_{ji} + 1} \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}$$

$$r^k_j = f_j(x^k) - \sum_{i=1}^{n} [P^k_{ji}(U^k_{ji} - x_i)^{s^k_{ji}} + Q^k_{ji}(x_i - L^k_{ji})^{s^k_{ji}}]$$

$$\max_j(L^k_{ji}) < \alpha^k_i \leq x_i^k \leq \beta^k_i < \min_j(U^k_{ji})$$

The tuning parameters $L^k_{ji}$ and $U^k_{ji}$ are lower and upper moving asymptotes, $s^k_{ji}$ are weighing factors, $\alpha^k_i$ and $\beta^k_i$ are lower and upper move limits, respectively. It is easy to prove that the approximate functions $f^k_j(x)$ are convex and separable for appropriate $L^k_{ji}$, $U^k_{ji}$ and $s^k_{ji}$. The approximate problem then is
minimize \[ f_0^k(x) = r_0^k + \sum_{i=1}^{n} [P_{0i}^k/(U_{0i}^k - x_i)^{\delta i} + Q_{0i}^k/(x_i - L_{0i}^k)^{\delta i}] \]

Subject to \[ f_j^k(x) = r_j^k + \sum_{i=1}^{n} [P_{ji}^k/(U_{ji}^k - x_i)^{\delta i} + Q_{ji}^k/(x_i - L_{ji}^k)^{\delta i}] \leq 0 \quad j = 1, 2, \ldots, m \]

and \[ \max\{\alpha_i^k, LB_i\} \leq x_i \leq \min\{\beta_i^k, UB_i\} \quad i = 1, 2, \ldots, n \quad (7) \]

The dual for this approximation problem is

\[ \text{maximize } W(\lambda) = r_0^k + \lambda^T r^k + \sum_{i=1}^{n} W_i(\lambda) \]

subject to \[ \lambda \geq 0 \quad (8) \]

where \[ r^k = (r_1^k, \ldots, r_m^k)^T, \]
and \[ \lambda = (\lambda_1, \ldots, \lambda_m)^T \]
is the vector of Lagrange multipliers or dual variables. The dual problem can be solved efficiently as it only has simple nonnegativity dual constraints. \[ W_i(\lambda) \]
can be obtained solving the following subproblems:

\[ W_i(\lambda) = \min_{x_i} \{ li(x_i, \lambda) \} \]

subject to \[ \max\{\alpha_i^k, LB_i\} \leq x_i \leq \min\{\beta_i^k, UB_i\} \quad (9) \]

where

\[ li(x_i, \lambda) = P_{0i}^k/(U_{0i}^k - x_i)^{\delta i} + Q_{0i}^k/(x_i - L_{0i}^k)^{\delta i} \]

\[ + \sum_{j=1}^{m} \lambda_j [P_{ji}^k/(U_{ji}^k - x_i)^{\delta i} + Q_{ji}^k/(x_i - L_{ji}^k)^{\delta i}] \]

The subproblems can be solved by line search to obtain the primal variables. The line search will add computational cost for problems with many design variables. However, since the subproblems are independent from each other for all \( i \), they can be solved in parallel.

In an alternative approach, let \( L_i^k = L_{ji}^k, U_i^k = U_{ji}^k \) and \( S_i^k = S_{ji}^k \) for all \( j \), i.e. lower and upper moving asymptotes as well as weighting factors
are selected to be the same for all functions but different for the design variables. The approximate functions now become

$$f_j^k(x) = r_j^k + \sum_{i=1}^{n} \left[ P_{ij}^k/(U_i^k - x_i)^{s_i} + Q_{ji}^k/(x_i - L_i^k)^{s_i} \right]$$  \hspace{1cm} (10)$$

where

$$P_{ji}^k = \begin{cases} 
(1/s_i^k)(U_i^k - x_i^k)^{(s_i^k+1)} \partial f_j(x^k)/\partial x_i & \text{if } \partial f_j(x^k)/\partial x_i > 0 \\
0 & \text{if } \partial f_j(x^k)/\partial x_i \leq 0 
\end{cases}$$

$$Q_{ji}^k = \begin{cases} 
0 & \text{if } \partial f_j(x^k)/\partial x_i > 0 \\
-(1/s_i^k)(x_i^k - L_i^k)^{(s_i^k+1)} \partial f_j(x^k)/\partial x_i & \text{if } \partial f_j(x^k)/\partial x_i \leq 0 
\end{cases}$$

$$r_j^k = f_j(x^k) - \sum_{i=1}^{n} \left[ P_{ij}^k/(U_i^k - x_i^k)^{s_i} + Q_{ji}^k/(x_i^k - L_i^k)^{s_i} \right]$$

$$L_i^k < x_i^k \leq x_i^k \leq \beta_i^k < U_i^k$$

The objective function of the subproblem in the dual problem is

$$l_i(x_i, \lambda) = (P_{0i}^k + \lambda^T P_i^k)/(U_i^k - x_i)^{s_i} + (Q_{0i}^k + \lambda^T Q_i^k)/(x_i - L_i^k)^{s_i}$$  \hspace{1cm} (11)$$

where \( P_i^k = (P_{1i}^k, \ldots, P_{mi}^k)^T \), \( Q_i^k = (Q_{1i}^k, \ldots, Q_{mi}^k)^T \). Now the subproblem can be solved explicitly. Let

$$l_i'(x_i, \lambda) = (1/s_i^k)[(P_{0i}^k - \lambda^T P_i^k)/(U_i^k - x_i)^{s_i^k + 1}$$

$$-(Q_{0i}^k + \lambda^T Q_i^k)/(x_i - L_i^k)^{s_i^k + 1}] = 0$$  \hspace{1cm} (12)$$

then

$$x_i = \frac{(Q_{0i}^k + \lambda^T Q_i^k)^{1/(s_i^k+1)}U_i^k + (P_{0i}^k + \lambda^T P_i^k)^{1/(s_i^k+1)}L_i^k}{(P_{0i}^k + \lambda^T P_i^k)^{1/(s_i^k+1)} + (Q_{0i}^k + \lambda^T Q_i^k)^{1/(s_i^k+1)}}$$  \hspace{1cm} (13)$$

and the primal design variables can be updated as follows

$$x_i^{k+1} = \begin{cases} 
\max\{\alpha_i^k, LB_i\} & \text{if } x_i \leq (\max\{\alpha_i^k, LB_i\}) \\
(x_i) & \text{if } (\max\{\alpha_i^k, LB_i\}) < x_i < (\min\{\beta_i^k, UB_i\}) \\
\min\{\beta_i^k, UB_i\} & \text{if } x_i \leq (\min\{\beta_i^k, UB_i\}) 
\end{cases}$$  \hspace{1cm} (14)$$
In the approximate functions Eq. (10), if \( s_i^k = 1 \) for all \( i \), then the method becomes the MMA whose approximate functions are

\[
f_j^k(x) = r_j^k + \sum_{i=1}^{n} [P_{ji}^k/(U_i^k - x_i) + Q_{ji}^k/(x_i - L_i^k)]
\]

where

\[
P_{ji}^k = \begin{cases} (U_i^k - x_i^k)^2 \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}
\]

\[
Q_{ji}^k = \begin{cases} 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ -(x_i^k - L_i^k)^2 \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}
\]

\[
r_j^k = f_j(x^k) - \sum_{i=1}^{n} [P_{ji}^k/(U_i^k - x_i^k) + Q_{ji}^k/(x_i^k - L_i^k)]
\]

\[
L_i^k < x_i^k \leq x_i^k \leq U_i^k < \alpha_i \leq x_i^k \leq \beta_i < U_i^k
\]

Further, in Eq. (15), if \( L_i^k = 0 \) and \( U_i^k = +\infty \) for all \( i \), then the method becomes the CONLIN one whose approximate functions are

\[
f_j^k(x) = r_j^k + \sum_{i=1}^{n} [P_{ji}^k x_i + Q_{ji}^k / x_i]
\]

\[
P_{ji}^k = \begin{cases} \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}
\]

\[
Q_{ji}^k = \begin{cases} 0 & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} > 0 \\ -(x_i^k)^2 \frac{\partial f_j(x^k)}{\partial x_i} & \text{if } \frac{\partial f_j(x^k)}{\partial x_i} \leq 0 \end{cases}
\]

\[
r_j^k = f_j(x^k) - \sum_{i=1}^{n} [P_{ji}^k x_i + Q_{ji}^k / x_i]
\]

\[
\alpha_i \leq x_i^k \leq \beta_i
\]

Thus, the approximation in Eq. (6) is gradually simplified into previous forms.
The approximation in Eq. (6) can be generalized further. Indeed any simple function of the primal design variables can be utilized as intermediate variable to construct an approximate problem. Different intermediate variables can be used in the same problem for different functions and variables. The only requirement is that the generated approximate problem is convex so that no duality gap exists. In both CONLIN and MMA, the approximation functions are separable with respect to $x_i$, making the dual problem easy to solve. A partially separable approximate problem can be also created, with intermediate variables that are simple functions of theprimals, and solved efficiently in the dual space. Lootsma [14] describes solution of a partially separable problem using a dual method. Complicated intermediate variables provide a relatively more accurate approximation but reduce the solution efficiency of the dual. The tradeoff is generally problem dependent. Flexibility in choosing intermediate variables offers an opportunity to exploit any special properties of the problem at hand.

Values for tuning parameters in convex approximation algorithms are also problem dependent in general. They can be determined empirically (e.g. Johanson and Papalambros [6]) and/or with some theoretical justification. Smaoui, Fleury and Schmit [7] use exact second order derivatives to determine moving asymptotes in MMA while Jiang and Papalambros [8] use second derivative approximations for that purpose. Move limits determine a trust region. If this region is large, the algorithm will move fast in the first few iterations and progress slowly close to the optimum, the converse happening if the trust region is small.

This concludes the discussion on generalizing the convex approximation methods. Suitable intermediate variables can be chosen to exploit special problem properties and to balance tradeoffs between accuracy of the approximation and efficiency in solving the dual problem.

CONVEX APPROXIMATIONS AND OPTIMALITY CRITERIA IN THE HOMOGENIZATION FORMULATION

It is shown that in the homogenization formulation of the topology problem the iteration scheme resulting from convex approximations is
the same as the one resulting from iterative solution of the primal optimality conditions.

In the approximate functions of Eq. (10), let \( L_i^k = 0 \) and \( U_i^k = +\infty \), as well as \( s_i^k = 1 \) if \( \partial f_j(x^k)/\partial x_i \geq 0 \) and \( s_i^k = s \) if \( \partial f_j(x^k)/\partial x_i \leq 0 \), for all \( i \). Then the approximate functions become

\[
 f_j^k(x) = r_j^k + \sum_{i=1}^{n} \left[ P_{ji}^k x_i + Q_{ji}^k / x_i^s \right]
\]  

(17)

where

\[
 P_{ji}^k = \begin{cases} \partial f_j(x^k)/\partial x_i & \text{if } \partial f_j(x^k)/\partial x_i > 0 \\ 0 & \text{if } \partial f_j(x^k)/\partial x_i \leq 0 \end{cases}
\]

\[
 Q_{ji}^k = \begin{cases} 0 & \text{if } \partial f_j(x^k)/\partial x_i > 0 \\ -(1/s)(x_i^k)^{s+1} \partial f_j(x^k)/\partial x_i & \text{if } \partial f_j(x^k)/\partial x_i \leq 0 \end{cases}
\]

\[
 r_j^k = f_j(x^k) - \sum_{i=1}^{n} \left[ P_{ji}^k x_i^k + Q_{ji}^k / (x_i^k)^{s} \right]
\]

This approximation is used now to solve the topology optimization model in Eq. (2). The derivatives of the objective (constraint) function with respect to design variables \( a \) and \( b \) are nonpositive (nonnegative)

\[
 \frac{\partial f_0(a, b, 0)}{\partial a} = -\sum_{(i,j,k,l=1)}^{2} \frac{1}{2} \int_{\Omega} \frac{\partial E_{ijkl}}{\partial a} \frac{\partial u_k}{\partial z_j} \frac{\partial u_l}{\partial z_i} d\Omega \leq 0
\]

\[
 \frac{\partial f_0(a, b, 0)}{\partial b} = -\sum_{(i,j,k,l=1)}^{2} \frac{1}{2} \int_{\Omega} \frac{\partial E_{ijkl}}{\partial b} \frac{\partial u_k}{\partial z_j} \frac{\partial u_l}{\partial z_i} d\Omega \leq 0
\]  

(18)

\[
 \frac{\partial f_1(a, b)}{\partial a} = 1 - b \geq 0 \text{ and } \frac{\partial f_1(a, b)}{\partial b} = 1 - a \geq 0
\]  

(19)

so the approximate problem is defined as

\[
 \text{minimize} \quad f_0^k(x) = r_0^k + \sum_{i=1}^{n} Q_{0i}^k / x_i^s
\]
subject to 

\[ f^k_i(x) = r^k_i + \sum_{i=1}^{n} P^k_{1i} x_i \leq 0 \]

and 

\[ \max \{ x_i^k, 0 \} \leq x_i \leq \min \{ \beta_i^k, 1 \} \] (20)

\[ r^k_0 = f_0(x^k) - \sum_{i=1}^{n} Q^k_{0i}/x_i^S \]

\[ r^k_1 = f_1(x^k) - \sum_{i=1}^{n} P^k_{1i} x_i \]

\[ Q^k_{0i} = (1/s)(x_i^k)^{s+1} \partial f_0(x^k)/\partial x_i \]

\[ P^k_{1i} = \partial f_0(x^k)/\partial x_i \]

with \( x = (a, b)^T \) being the design variables. The orientation angles \( \theta \) can be updated as previously using the directions of the principle stresses.

The dual problem is now

\[ \text{maximize } W(\lambda) = r^k_0 + \lambda r^k_1 + \sum_{i=1}^{n} W_i(\lambda) \]

subject to 

\[ \lambda \geq 0 \] (21)

with only one dual variable (Lagrange multiplier) \( \lambda \). The dual is easily solved by updating \( \lambda \) to satisfy \( \partial W(\lambda)/\partial \lambda = f^k_i(x) = 0 \), similarly to the inner loop in the optimality criteria method.

The subproblem for the primal variables is

\[ W_i(\lambda) = \min_{x_i} \{ l_i(x_i, \lambda) \} \]

\[ l_i(x_i, \lambda) = Q_{0i}/x_i^S + \lambda P_{1i} x_i \]

subject to 

\[ \max \{ x_i^k, 0 \} \leq x_i \leq \min \{ \beta_i^k, 1 \} \] (22)
and it can be solved explicitly. Let

\[ \ell_i'(x, \lambda) = -sQ_{0i}/x_i^{s+1} + \lambda P_{0i}^k = 0 \]

then

\[ x_i = \left( \frac{sQ_{0i}}{\lambda P_{0i}^k} \right)^{1/(s+1)} = x_i^k \left( -\frac{\partial f_i(x^k)/\partial x_i}{\lambda \partial f_i(x^k)/\partial x_i} \right)^{1/(s+1)} \]  \hspace{1cm} (23)

Letting \( \eta = 1/(s+1) \) and substituting Eqs. (18), (19) and (4) into Eq. (23) gives the expressions for the design variables. For \( a \) the update formula becomes

\[ a^{k+1} = \begin{cases} 
(\max\{x_i^k, 0\}) & \text{if } a^k(D_a^k)\eta \leq \max\{x_i^k, 0\} \\
(a^k(D_a^k)^\eta) & \text{if } \max\{x_i^k, 0\} \leq a^k(D_a^k)^\eta \leq \min\{\beta_i^k, 1\} \\
(\min\{\beta_i^k, 1\}) & \text{if } \min\{\beta_i^k, 1\} \leq a^k(D_a^k)^\eta
\end{cases} \]  \hspace{1cm} (24)

and similarly for \( b \). Letting \( \alpha^k = (1 - \zeta)a^k \) and \( \beta^k = (1 + \zeta)a^k \), Eq. (24) becomes identical to Eq. (3).

The iteration scheme derived from a convex approximation method is then the same as that derived using optimality criteria. The only difference is in solving the dual problem of Eq. (21). The primal model in Eq. (2) has a single constraint which is active and inexpensive to compute. In the optimality criteria method the multiplier is updated by solving this primal constraint, \( f_i(x) = 0 \). In the convex approximation method the multiplier is updated by solving the approximate constraint, \( f_i^*(x) = 0 \). Note, however, that \( f_i(x) \) contains only linear and bilinear terms and its linear approximation \( f_i^*(x) \) is a very good one.

When several constraints are included in the primal, the generalized approximation algorithm remains unchanged, except that now there will be several dual variables. However, in solving the dual, only the approximate problem is involved and no structural reanalysis is needed. Structural analyses constitute the majority of computational cost during optimization and solution of a dual with several variables will not significantly affect the efficiency of the overall optimization process compared to the single variable case. Thus, a topology
optimization model with multiple constraints can be solved with a convex approximation algorithm with a relatively small added cost over the single constraint model.

APPLICATION TO MULTIPLE LOAD EXAMPLES

A *multipurpose* topology optimization problem attempts to obtain an optimal topology of a structure that functions under multiple load cases, i.e. loads are not applied at the same time but one by one as different load conditions. Such a problem was solved with homogenization by Diaz and Bendsøe [15] using a weighted mean compliance for all load cases, and by Fukushima, Susuki and Kikuchi [16] using the maximum value of compliance in each element among all load cases. A variety of models for multipurpose topology optimization can be solved using the convex approximation methods described above. The details are omitted here and only a simple model is presented to generate examples of multi-constraint models.

The optimization model used here is to minimize the volume of the structure subject to its mean compliance being less than a given mean compliance for all load cases.

Assuming the equilibrium equations are satisfied by solving the finite element equations, the model becomes

Minimize \( f_0(x) = v(x) \)

subject to \( f_j(x) = C_j(x) - C_j \leq 0 \) \( j = 1, \ldots, m \) \hspace{1cm} (25)

where \( x \) is a vector of design variables \( a \) and \( b \), \( v(x) \) is the same volume as \( f_1(a, b) \) in Eq. (1), \( C_j(x) \) are mean compliances expressed in the same manner as \( f_0(a, b, \theta) \) in Eq. (1) for each load case, \( C_j \) is the given mean compliance, and \( m \) is the number of load cases.

Intermediate variables are chosen to be \( x_i \) for the volume function and \( 1/x_i \) for the mean compliance functions. The approximate convex problem is

\[
\text{minimize} \quad f_0^k(x) = r_0^k + \sum_{i=1}^{n} P_{0i}^k x_i
\]
subject to $f_j^k(x) = r_j^k + \sum_{i=1}^{n} Q_{ji}^k/x_i^s \leq 0 \quad j = 1, \ldots, m$

max\{x_i^k, 0\} \leq x_i \leq \min\{\beta_i^k, 1\} \quad (26)$

$$r_j^k = f_j(x) - \sum_{i=1}^{n} Q_{ji}^k/x_i^s$$

$$r_0^k = f_0(x) - \sum_{i=1}^{n} P_{0i}^k x_i$$

$$P_{0i}^k = (1/s)(x_i^k)^{s+1} \delta f_0(x)/\delta x_i$$

$$Q_{ji}^k = \delta f_j(x)/\delta x_i$$

The dual problem is

maximize $W(\lambda) = r_0^k + \lambda^T r^k + \sum_{i=1}^{n} W_i(\lambda)$

subject to $\lambda \geq 0 \quad (27)$

where $r^k = (r_1^k, \ldots, r_m^k)^T$, and $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is the vector of dual variables. The dual problem can be solved by a standard SQP algorithm with only bound nonnegativity constraints. (A tailored Quasi-Newton or conjugate gradient method modified for simple bound constraints may be somewhat more efficient, if readily available).

The subproblem is

$$W_i(\lambda) = \min_{x_i} \{\text{li}(x_i, \lambda)\}$$

$$\text{li}(x_i, \lambda) = P_{0i}^k x_i + \sum_{j=1}^{m} \lambda_j Q_{ji}^k/x_i^s$$

subject to max\{x_i^k, 0\} \leq x_i \leq \min\{\beta_i^k, 1\} \quad (28)$

An explicit solution is obtained by letting

$$\text{li}'(x_i, \lambda) = P_{0i}^k - \sum_{j=1}^{m} (\lambda_j s Q_{ji}^k/x_i^{s+1}) = 0$$
and then

\[
x_i = \left[ \sum_{j=1}^{m} \left( \lambda_j sQ_{ij}^k / P_{0i}^k \right) \right]^{1/(s+1)}
= x_i \left\{ 1 - \sum_{j=1}^{m} \left[ \lambda_j \frac{\partial f_j(x^k)}{\partial x_i} \right] \left[ \frac{\partial f_0(x^k)}{\partial x_i} \right] \right\}^{1/(s+1)}
\]

(29)

Therefore, the design variables \( x \) can be updated by the formula

\[
x_i^{k+1} = \begin{cases} 
\max\{\alpha_i^k, 0\} & \text{if } x_i \leq \max\{\alpha_i^k, 0\} \\
x_i & \text{if } \max\{\alpha_i^k, 0\} < x_i < \min\{\beta_i^k, 1\} \\
\min\{\beta_i^k, 1\} & \text{if } \min\{\beta_i^k, 1\} \leq x_i 
\end{cases}
\]

(30)

where we let \( \alpha_i^k = (1 - \xi)x_i^k \) and \( \beta_i^k = (1 + \xi)x_i^k \).

The design variables \( \theta \) can be treated separately as in Diaz and Bendsøe [15]. The Lagrange multipliers \( \lambda \) are equivalent to the weighting factors in the weighted compliance formulation of Diaz and Bendsøe. Two examples are solved next using the above formulation.

**Example 1** Cantilever with two load cases

The design domain is a \( 1 \times 1 \) area with the left edge fixed with full material between the supports, Figure 1. The structure is required to support the vertical shear force \( F_1 = 1 \) at the middle of the right edge or the horizontal tension force \( F_2 = 1 \) at the upper right corner. The domain is divided into 2500 elements and 50 elements are in non-designable domain so there are 4900 design variables, because the orientation angles \( \theta \) are computed separately.

First, it was assumed that only \( F_1 \) is applied to the structure and this load case was solved with the optimality criteria method using the model in Eq. (1). The given volume constant was 0.3, the weighting factor \( \eta \) was 0.8, and the move limit \( \xi \) was 0.3. After 25 iterations the optimal topology shown in Figure 3 was obtained with optimal mean compliance of 0.19. Next it was assumed that forces \( F_1 \) and \( F_2 \) are applied one by one, and the multipurpose problem was solved by the convex approximation method described in Eq. (26)–(30) using the model in Eq. (25). The given mean compliances used were \( C_1 = 0.21 \) and \( C_2 = 0.16 \), while \( \xi = 0.03 \) and \( s = 0.25 \) (equivalent to \( \eta = 0.8 \)). After
FIGURE 2  Example 1: Cantilever with two load cases.

FIGURE 3  Solution of Example 1 with single load.
25 iterations the optimal topology shown in Figure 4 was obtained with optimal volume of 0.33.

Comparing the observed efficiencies is interesting. Using a standard workstation, in each iteration the finite element analysis takes about 0.45 minutes, and the optimization calculations take about 0.06 minutes in the optimality criteria method (one constraint) and 0.12–0.15 minutes in the convex approximation method (two constraints).

Example 2 Structure with two fixed ends and three load cases
The design domain is a $1 \times 2$ area with left and right bottom corners fixed, Figure 5. The loads $F_1 = F_2 = F_3 = 1$ are equally spanned and applied at the top edge. The domain is divided into 3200 elements so there are 6400 design variables (orientation angles $\theta$ are computed separately).

First the three loads were assumed to be applied simultaneously and the single purpose problem was solved with the optimality criteria method using the model in Eq. (1). The volume constant was 0.6,
$\eta = 0.8$, and $\xi = 0.3$. After 25 iterations the optimal topology shown in Figure 6 was obtained with optimal mean compliance of 0.42. Next the multipurpose problem was solved with the convex approximation method using the model of Eq. (26). The given mean compliance was $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = 0.1$. After 25 iterations the optimal topology shown in Figure 7 was obtained with optimal volume of 0.66. Here again, in each iteration the finite element analysis takes about 0.44 minutes, and the optimization calculations take about 0.08 minutes in the

![Diagram](image_url)

**FIGURE 5** Example 2: Structure with two fixed ends and three load cases.

![Diagram](image_url)

**FIGURE 6** Solution of Example 2 with all loads acting simultaneously.
optimality criteria method and 0.17–0.24 minutes in the convex approximation method (three constraints).

CONCLUDING REMARKS

The generalization of convex approximation methods presented here is very appealing, not only for the homogenization-based topology problems that motivated it, but also as an efficient algorithm for certain classes of large scale problems. Although no formal convergence analysis has been performed, it is expected that convergence rates will be at least linear. Computational costs appear to increase in direct proportion to the number of constraints in the primal although this performance will depend on the efficiency of the dual solver. The class of problems for which such convex approximations should prove successful includes models with many functions monotonic with respect to the design variables, a property preserved by the approximating problems.

For topology design, the techniques described above substantially expand the utility of the homogenization-based approach. They allow a more direct and natural statement of the design problem and a more direct way of handling multiple load cases. Moreover, in an integrated structural design system (e.g. Chirehdast and Papalambros [17], Bremicker et al., [18], Chirehdast et al., [19]) some of the constraints that were previously left for a subsequent detailed design phase can be
now included in the initial design phase where the topology or layout of the structure is first created. A more complete study of design tradeoffs can be accomplished early.

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