A NOTE ON WEIGHTED CRITERIA METHODS FOR COMPROMISE SOLUTIONS IN MULTI-OBJECTIVE OPTIMIZATION

TIMOTHY WARD ATHAN\(^1\) and PANOS Y. PAPALAMBROS\(^2\)

\(^1\)General Motors Corporation, Powertrain Group, Willow Run Plant, Mail Code 732 Ypsilanti, MI 48198-6198, U.S.A.
\(^2\)Department of Mechanical Engineering and Applied Mechanics University of Michigan, Ann Arbor, Michigan 48109-2125, U.S.A

(Received 3 April 1995; in final form 20 March 1996)

A common multi-objective optimization approach forms the objective function from linearly weighted criteria. It is known that the method can fail to capture Pareto optimal points in a non-convex attainable region. This note considers generalized weighted criteria methods that retain the advantages of the linear method without suffering from this limitation. Compromise programming and a new method with exponentially weighted criteria are evaluated. Demonstration on design problems is included.

Keywords: Multicriteria; multiobjective; design optimization; non-convex; Pareto solutions

INTRODUCTION

Multicriteria or multi-objective optimization is the search for a solution that best manages trade-offs between criteria that conflict and that cannot be converted to a common measure.

DEFINITION 1 The minimization formulation of the multicriteria optimization (MCO) problem is

\[
\text{Minimize } c(x, b) \\
\text{subject to}
\]

(1)

155
\[ g(x, b) \leq 0 \]
\[ h(x, b) = 0 \]

Here \( c \) is the vector of \( I \) real valued criteria \( c_i \). The design parameters, \( b \), are not controlled by the decision maker. From the design variables, \( x \), are those that satisfy the constraints \( g(x, b) \) and \( h(x, b) \) and they are termed the feasible set, \( F \). The corresponding values for \( c(x, b) \) constitute the attainable set, \( A \).

A range of methods is available to convert the multicriteria formulation into a substitute problem with a scalar objective that can be solved with the tools of single objective optimization.

**Definition 2** The scalar substitute problem for a multicriteria optimization problem is

\[
\text{Minimize} \quad f(c, M) \\
\text{subject to} \\
\quad g(x, b) \leq 0 \\
\quad h(x, b) = 0
\]

with \( f \) a scalar valued function, and \( M \) a set of parameters (weights or other factors) which are adjusted by the decision maker to tune the objective function to match the decision maker's preferences. These parameters will be referred to as preference parameters.

In the weighted criteria method a multiplicative weight is selected for each criterion and the summation of all weighted criteria is the new, single objective to be minimized.

**Definition 3** The (linearly) weighted criteria problem is

\[
\text{Minimize} \quad f(c) = w_1c_1(x, b) + w_2c_2(x, b) + \cdots + w_Ic_I(x, b) \\
\text{subject to} \quad g(x, b) \leq 0 \\
\quad h(x, b) = 0
\]
\[ \sum w_i = 1 \quad w_i \geq 0 \]

The \( w_i \) are scalar-valued weights, with values from 0 to 1.

The weighted criteria problem may be solved repeatedly with different sets of weights. In this manner the decision maker learns about available trade-offs and is offered a selection of candidate solutions. This may be the most popular MCO method; the use of weights to represent relative preferences between the criteria is direct and intuitive, the formulation is simple and easy to use, and solutions are Pareto optimal.

**Definition 4** [Edgeworth [1], Pareto [2]] A point \( c^0 \) in the attainable set \( \mathcal{A} \) is Pareto optimal if and only if there is not another \( c \in \mathcal{A} \) such that \( c_i \leq c_i^0 \) for all \( i \) and \( c_i < c_i^0 \) for at least one \( i \).

In the next section of this paper the performance of the linear weighted criteria method is examined. The scope of the evaluation is then enlarged to include other weighted criteria objective functions. The desirable properties of a weighted criteria objective function are formulated. Two weighted criteria methods are then considered: weighted compromise programming (WCP) and a new exponential weighted criteria method. It is shown that both of these methods possess the desired qualities.

The fourth section consists of three design demonstrations of these two methods. A fifth section presents concluding remarks. Any proposition or proof presented in this article without attribution is original.

**Performance of the Weighted Criteria Method**

**Theorem 1** [Stadler [3]] Solution of a scalar objective function is sufficient for Pareto optimality if the objective function increases monotonically with respect to each criterion.

*Proof* [See Athan [4]].

**Proposition 1** All Pareto optimal points lie upon the boundary of the attainable set.

*Proof* [See Lin [5]].
Convexity and Attainable Sets

Attainable set convexity can have important ramifications for MCO methods. One useful definition of convexity is $\Lambda^>$ convexity. Here $\Lambda^> = \{ c \in \mathbb{R}^I | c_i \geq 0 \text{ for } i = 1, \ldots, I \}$ represents the positive cone.

DEFINITION 5 [Yu [6]] The set $A$ is $\Lambda^>$ convex if and only if $A + \Lambda^>$ is a convex set.

PROPOSITION 2 Solution of the weighted criteria method problem is sufficient for Pareto optimality.

Proof [See Athan [4]].

Conditions for Solution to be Necessary for Pareto Optimality

The weighted criteria method forms an objective function that is an affine combination of the competing criteria. Each constant objective value surface is a hyperplane in criteria space,

$$K = w_1 c_1 + w_2 c_2 + \ldots + w_I c_I$$

(4)

for an objective function value of $K$.

PROPOSITION 3 For any scalar substitute objective function for a multicriteria optimization problem with a continuous and smooth attainable region boundary near a Pareto optimal point, $c^0$, the optimization with respect to the objective function will have solution $c^0$ only if the attainable region boundary and the objective function constant value surfaces are tangent at $c^0$.

Proof [See Athan [4]].

![Diagram](image)

FIGURE 1 Constant objective lines of the weighted criteria method.
PROPOSITION 4 [Yu [6]] If the attainable set is $\Lambda^\omega$ convex then minimization of the weighted criteria scalar objective function is necessary for Pareto optimality.

Proof [See Yu [6]].

Non-Convex Attainable Sets

In this section it will be shown that the weighted criteria method can have a limitation when used with non-convex attainable sets.

PROPOSITION 5 [Yu [6]] $c^0$ is Pareto optimal if for any $i$ between one and $I$, $c^0$ uniquely minimizes $c_i$, for all $c \in C_i(c^0) = \{c \in A | c_j \leq c^0_j, j \neq i, j = 1, \ldots, I\}$.

Proof [See Yu [6]].

PROPOSITION 6 [Uncertain origin, stated in Osyczka [7] and in Vincent and Grantham [8]]. If the Pareto set is not $\Lambda^\omega$ convex then solution of the weighted criteria scalar substitute problem is no longer a necessary condition for Pareto optimality.

Proof [See Athan [4]].

Proposition 6 indicates that in some cases not all Pareto optimal points will be found by the weighted criteria method. One example is depicted in Figure 2.

PROPOSITION 7 [Yu 6] If each $c_i(x)$, $i = 1, \ldots, I$ is convex with respect to $x$ on a convex feasible set $F$, then $A = \{c(x)| x \in F\}$, where $c = (c_1, \ldots, c_I)$, is $\Lambda^\omega$ convex.

Proof [See Yu [6]].

Koski suggests that non-convex attainable regions may be common in structural optimization [9]. Two examples discussed below are provided in that reference. Other examples have been published by Baier [10], and by Stadler and Dauer [11].

The existence of Pareto optimal points on a non-convex section of the attainable set boundary should be suspected when a small change in the weights used for the weighted criteria method results in a large change in the solution. This phenomenon could occur also if the attainable set is
non-connected. The conditions for a non-connected but otherwise convex set have been determined by Naccache [12].

OTHER WEIGHTED CRITERIA METHODS

A generalized weighted criteria scalar substitute problem is considered:

**Definition 6** The generalized weighted criteria MCO scalar substitute objective function is

\[ f = \sum_{i=1}^{I} f_1(w_i) f_2(c_i, m_i). \]  

Scalar \( w_i \) and vector \( m_i \) are the preference parameters that can be adjusted by the decision maker. To adhere to the structure of the established weighted criteria method the objective function is separable by criteria, and the weights, \( w_i \), will be variable between 0 and 1 and will sum to 1.

The determination of appropriate functions in Eq. (5) above is the goal of this section. The definition is two steps removed from the definition of the linear weighted criteria method. Intermediate is the sum of a single
function of the variables, \( f = \sum_{i=1}^{I} f(w_i, c_i, m_i) \). The variables reflect different types of parameters and therefore it is easier to develop formulations with their functions explicitly differentiated, as in the definition of the generalized weighted criteria method.

**Desirable Properties**

The desirable properties for an MCO substitute problem objective function are:

**Property 1** Solution of a scalar optimization with the objective function should be sufficient for Pareto optimality. By Theorem 1, the function should therefore be an increasing function with respect to every criterion.

**Property 2** Solution of a scalar optimization with the objective function should be necessary for Pareto optimality. Any Pareto optimal point should be obtainable as a solution by adjusting the function's parameters.

**Property 3** The function should be easy to understand and use.

**Property 4** The potential for numerical problems such as overflow and underflow should be small.

**Candidate Functions**

To be an increasing function with respect to each criterion as required by Property 1 each term must have a positive first derivative with respect to the criterion in that term. Property 2 will not be realized unless the curvatures of the constant value curves are negative. This requires a negative definite Hessian matrix. More than two adjustment parameters for each term in the objective function may make the use of the method unwieldy, a consideration relevant to Property 3.

These considerations suggest the formulations described in Definitions 7 and 8 below. The definitions contain unspecified functions \( f_1 \) and \( f_2 \) that will be specified in Definitions 10 and 11 later in this section.

**Definition 7** The exponential weighted criteria scalar substitute objective function with unspecified weighting function \( f_1 \) is

\[
f = f_1(w_1) \exp(m_1 c_1) + f_1(w_2) \exp(m_2 c_2) + \ldots f_1(w_I) \exp(m_I c_I)
\] (6)
More generally, each term in the exponential weighted criteria objective could have the form $f_j(w_i)P^{mc_i}$ for any constant $P$, but selecting $P=e$ simplifies computations.

**Definition 8** The weighted compromise programming (WCP) scalar substitute objective function with unspecified weighting function $f_2$ is

$$f = f_2(w_1)c_1^{-m_1} + f_2(w_2)c_2^{-m_2} + \cdots + f_2(w_i)c_i^{-m_i} \quad \text{with scaled } c_i \geq 0 \quad (7)$$

Compromise programming was originally proposed by Charnes and Cooper for linear optimization problems [10], and was later extended to nonlinear ones by Dauer and Krueger [11]. The exponential weighted criteria method can be regarded as a special case of WCP, as substitution of $\tilde{c}_i = e^{ct_i}$ into Eq. (6) delivers Eq. (7).

If the criteria were not restricted to non-negative values in this formulation then the function would not be an increasing function with respect to those criteria with negative values when the $m_i$ are even integers. Also, negative criteria values would generate positive curvatures for the constant objective values surfaces. The use of different values for the $m_i$ is an unnecessary complication. It would offer a scaling between the terms, but this can be adequately effected with the weights $w_i$ and criteria scaling before optimization. There is no capability that differing $m_i$ would confer that cannot be realized with scaling and the criteria weights. The parameters $m_i$ can therefore be set to the single parameter $m$.

Constant value curves for both functions with two values for $m$ are presented in Figures 3 through 6. In all plots $f_{1or2}(w_1) = f_{1or2}(w_2) = 0.5$. As $m$ is increased the curves more closely approximate a negative cone $\Lambda \leq$ that has been translated.

**Determination of Weighting Functions $f_1$ and $f_2$**

The unspecified weighting functions in the objective function definitions will now be determined. By Property 3 the method should be easy to use, and this objective supports a weighting function that responds to decision maker input in an easily understood manner. One measure of changes to the constant objective function surfaces is the ratio of the positive axis intercepts, $c_2^{int}/c_1^{int}$. 
DEFINITION 9 The axis intercept, $c_i^{\text{int}}$, is the value of $c_i$ at the intersection of the $c_i$ axis and an objective function constant value surface. The axis intercept $c_i^{\text{int}}$ is determined by setting all criteria but $c_i$ to zero in the equation for an objective function constant value surface, and solving for $c_i$.

For the linear weighted criteria objective function the ratio of intercepts is

$$c_j^{\text{int}} / c_i^{\text{int}} = w_i / w_j$$  

(8)
This linear relationship helps to make the method easy to use. To obtain the same relationship for the WCP method, $f_1(w_i)$ is set to $w_i^m$.

**Definition 10** The weighted compromise programming (WCP) scalar substitute objective function is

$$f = (w_1c_1)^m + (w_2c_2)^m + \cdots + (w_ic_i)^m$$  \hspace{1cm} (9)
Because the weights \( w_i \) lie between 0 and 1, this selection for \( f_1 \) helps reduce the likelihood of overflow problems.

The exponential weighted criteria function’s ratio of intercepts cannot be made to correspond exactly to that of the linear weighted criteria objective function. A linear relationship between the weights and the intercepts is desired. Additionally, it is desirable that \( f_2(0) = 0 \), for this provides the decision maker with a convenient way to select preference parameters to make the objective function independent of one or more criteria. These considerations suggest \( f_2(w_i) = (\exp(mw_i) - 1) \).

**Definition 11** The exponential weighted criteria scalar substitute objective function is

\[
f = (\exp(mw_1) - 1)\exp(m_c_1) + (\exp(mw_2) - 1)\exp(m_c_2) + \cdots + (\exp(mw_I) - 1)\exp(m_c_I)
\]  

(10)

The ratio of the intercepts is

\[
c_j^{\text{int}}/c_1^{\text{int}} = \left[ \ln \left( K - I + 1 - \sum_{i=1}^{I} \exp(mw_i) + \exp(mw_j) \right) \right] / \left[ \ln \left( K - I + 1 - \sum_{i=1}^{I} \exp(mw_i) + \exp(mw_1) \right) \right]
\]

(11)

Here \( K \) is the value of the objective function on the constant value curve. In Eq. (11) the intercept ratio is a logarithmic function of exponential functions, suggesting a relation similar to the linear relation of Eq. (8). One drawback of this selection for \( f_2 \) is that it increases the susceptibility to overflow problems.

**Analysis of the Candidate Functions**

The two modifications to the weighted criteria objective function formulation are analyzed in this section for their possession of the desirable properties listed above. By design the solution of either formulation is a sufficient condition for Pareto optimality, satisfying Property 1. Also they were designed for ease of use, Property 3.
There are two parts to the proof that solution of a scalar optimization with the objective function is necessary for Pareto optimality. By Proposition 2, a set of preference parameters must exist for each Pareto optimal point such that the objective function constant objective surface is tangent to the attainable region at the point. Secondly, for each Pareto optimal point a set of preference parameters must exist such that the value of the objective function at the point is less than at any other attainable point.

**Proposition 8** Solution of the weighted compromise programming (WCP) scalar substitute problem is necessary for Pareto optimality.

*Proof* The first derivatives take the form

$$\frac{\partial c_I}{\partial c_i} = - \left( \frac{w_i}{w_I} \right)^m \left( \frac{c_i^0}{c_I^0} \right)^{m-1} \text{ for } i = 1 \text{ to } I - 1$$  \hspace{1cm} (12)

The first condition requires that these derivatives are equal to the gradient of the attainable region at the Pareto optimal point. At a Pareto optimal point all of the partial derivatives for the attainable region surface must be negative, for otherwise both $c_i$ and $c_I$ could be reduced without changing other criteria values. These $I - 1$ equations must be met using the $I$ adjustable weights. Eq. (12) is monotonic with respect to the weights, so the adjustments are possible. A lower bound, $M^0$, will be imposed by the second half of the proof. Because there is freedom to select any $M$ greater than $M^0$, a value can be selected large enough so that the weights, $w_i$, can be set such that they will sum to one.

Now it must be shown that these weights ensure that $f(c) > f(c^0)$ for each Pareto optimal point $c^0$ and for every attainable point, $c$.

$$f(c) - f(c^0) = w_1^m (c_1^m - (c_1^0)^m) + w_2^m (c_2^m - (c_2^0)^m)$$

$$+ w_3^m (c_3^m - (c_3^0)^m)$$

$$+ \ldots + w_I^m (c_I^m - (c_I^0)^m)$$  \hspace{1cm} (13)
WEIGHTED CRITERIA METHODS

The tangency requirement imposes

\[
\left( \frac{w_i^m}{w_I^m} \right)^{m-1} \left( \frac{c_i^0}{c_I^0} \right)^{m-1} = L_i > 0 \quad \text{for } i = 1 \text{ to } I - 1 \tag{14}
\]

Solving for \( w_i^m \), substituting into Eq. (13), dividing out \((c_i^0)^{m-1}\) \( w_i^m \) because it is always positive, and seperating numerator terms, provides

\[
f(c) - f(c^0) = L_1 \left( \left( \frac{c_i^0}{c_i^0} \right)^{m-1} c_i - c_i^0 \right) + L_2 \left( \left( \frac{c_2^0}{c_2^0} \right)^{m-1} c_2 - c_2^0 \right) + \ldots + \left( \left( \frac{c_I^0}{c_I^0} \right)^{m-1} c_I - c_I^0 \right) \tag{15}
\]

The \( c_i \)'s are all positive by definition, and the \( L_i \)'s are positive also. Because \( c^0 \) is Pareto optimal at least one of the fractions must be larger than one. As \( m \) increases that term will dominate all negative terms, making Eq. (15) positive. \( \square \)

**Proposition 9** Solution of the exponential weighted criteria scalar substitute problem is necessary for Pareto optimality.

**Proof** The first derivatives take the form

\[
\frac{\partial c_i}{\partial c_i} = -\left( \frac{\exp(mw_i) - 1}{\exp(mw_i) - 1} \right) \exp(m(c_i^0 - c_i^0)) \quad \text{for } i = 1 \text{ to } I - 1 \tag{16}
\]

The first condition requires that these derivatives be equal to the gradient of the attainable region boundary at the Pareto optimal point. At a Pareto optimal point all partial derivatives for the attainable region boundary must be negative, for otherwise criteria could be reduced without increasing other criteria. These \( I - 1 \) equalities must be met with \( I \) adjustable weights. Equations (16) are monotonic with respect to the weights, so the adjustments are possible. A lower bound on \( m, M^0 \), will be imposed by the second half of the proof. Because there is freedom to select any \( m \) greater than \( M^0 \), a value can be selected large enough so that the weights, \( w_i \), can be set such that they will sum to one.
Now it must be shown that these weights ensure that \( f(c) > f(c^0) \) for each Pareto optimal point \( c^0 \) and for every attainable point \( c \).

\[
f(c) - f(c^0) = (\exp(mw_1 - 1))(\exp(mc_1) - \exp(mc^0_1)) \\
+ (\exp(mw_2 - 1))(\exp(mc_2) - \exp(mc^0_2)) \\
+ (\exp(mw_3 - 1))(\exp(mc_3) - \exp(mc^0_3)) \\
+ \cdots + (\exp(mw_I - 1))(\exp(mc_I) - \exp(mc^0_I))
\] (17)

Using the tangency requirement

\[
\frac{(\exp(mw_i - 1))}{(\exp(mw_I - 1))} \exp(m(c^0_i - c^0_I)) = L_i \quad \text{for} \quad i = 1 \quad \text{to} \quad I - 1
\] (18)

cancelling \( (\exp(mw_I - 1)) \) because it will always be positive for large \( m \), multiplying exponential terms, and dividing out \( \exp(mc^0_I) \) because it is always positive, Eq. (17) results in

\[
f(c) - f(c^0) = L_1 \left( \exp(m(- c^0_1 + c_1)) - 1 \right) + L_2 \\
\left( \exp(m(- c^0_2 + c_2)) - 1 \right) + L_3 \left( \exp(m(- c^0_3 + c_3)) - 1 \right) \\
+ \cdots + \left( \exp(m(- c^0_I + c_I)) - 1 \right)
\] (19)

All \( L_i \) are positive. Because \( c^0 \) is Pareto optimal at least one of the exponential terms will be positive. That term will dominate Eq. (19) when \( m \) is large, ensuring a positive sum. □

Susceptibility to overflow and underflow is the subject of Property 4. Large values of \( m \) could lead to numerical overflow with the exponential weighted criteria method and to underflow in the WCP method. These can be prevented by multiplying the objective functions by scaling factors. Scaling can be made automatic within the optimization code.

**Summary**

Both methods are useful alternatives to the established linear method. Neither shares the linear method’s limitation with non-convex attainable regions.
The WCP method was defined with all criteria restricted to non-negative values, and this is a limitation. Negative criteria values can be prevented by a shift of datum. A large shift will ensure non-negative values, but could result in large criteria values. Experience with a particular design problem will provide an estimate of minimum criteria values. If the minimum is known for each criterion they can be used together as an ideal point, and this would eliminate the concern about negative criteria.

Normalization of the criteria before their use in these methods will provide a more evenly distributed solution set. Estimates of minimum criteria values can be used for normalization.

EXAMPLES

The new scalar objective function formulations are demonstrated in three examples. The first two examples are from Koski's paper [9].

Example 1  A static three-bar truss problem [Koski [9]]

A three-bar truss under one static loading condition is Koski's first example. The total volume of the truss and a linear combination of the two nodal displacements are to be minimized. The design variables are the three cross sectional areas of the members. Stress and side constraints are imposed. The three bar truss is shown in Figure 7.

Numerical data are as follows: \( F = 20 \text{ kN}, L = 100 \text{ cm}, \) Young's Modulus \( E = 200 \text{ GPa}, \) bar tension limit \( \sigma_T = 200 \text{ MPa}, \) bar compression limit \( \sigma_C = -200 \text{ MPa}, \) bar cross section area upper limit \( A_U = 2 \text{ cm}^2, \) bar cross section area lower limit, \( A_L = 0.1 \text{ cm}^2. \) The criteria are: criterion \( c_1 = \) volume of the truss bars, \( V; \) criterion \( c_2 = 0.25 \) times the vertical displacement, \( d_{v}, \) of the lowest node plus 0.75 times the horizontal displacement, \( d_{h}, \) of the lowest node, \( \Delta. \) The problem statement is:

\[
\text{Minimize } \{ V(A_1, A_2, A_3), \Delta (A_1, A_2, A_3) \} \\
\text{subject to } A_L \leq A_i \leq A_U, i = 1, 2, 3 \\
\sigma_C \leq \sigma \leq \sigma_T, i = 1, 2, 3
\]

(20)
For small displacements the tension $T_i$ in the three bars, numbered left to right, can be approximated as: $T_1 = A_1 E (d_c \cos(45°) - d_h \cos(45°))/\sqrt{2L}$, $T_2 = A_2 E d_v/L$, $T_3 = A_3 E (d_c \cos(60°) + d_h \cos(30°))/(2L)$. Then the vertical force balance is $F = T_2 + T_1 \cos(45°) + T_3 \cos(60°)$, and the horizontal, $F = T_3 \sin(60°) - T_1 \sin(45°)$. Combining these equations the two unknown displacements can be determined. Dividing by respective bar cross sectional areas gives the stresses. The volume equals $\sqrt{2LA_1 + LA_2 + 2LA_3}$.

The attainable set is shown in Figure 8. Points upon boundary segment AB and boundary segment CD are Pareto optimal, while points upon segment BC are not. The Pareto optimal points on the non-convex boundary AB correspond to combinations of two of the design variables while the third is held constant at its upper bound. Were there no stress constraints the attainable region would include some designs with volume lower than shown in the figure. The new lower boundary would be entirely convex and would dominate the Pareto optimal points that lie on curves AB and CD. Thus even though the constraints remove points adjacent to the convex boundary CD they play a key role in the existence of Pareto optimal points on a non-convex boundary in this problem.

Theoretically, the linear weighted criteria method should find only point A from the entire Pareto segment AB, as well as all of Pareto
segment CD. In practice, as part of the present study and using a sequential quadratic programming optimizer [12], some points adjacent to A were found as well, because curvature there is small. The rest of segment AB was missed.

All Pareto optimal points could be found with the alternative weighted criteria methods. The exponential formulation found nearly all of the full set using an $m$ value of one. Point $B$ and the points slightly above $B$ required an $m$ value of two. With the WCP method an $m$ value of 5 was large enough for all Pareto optimal points to be found.

**Example 2  A dynamic two-bar truss problem** [Koski [9]]

Koski's second example considers the dynamic behavior of a two-bar truss. The total volume of the structure and the lowest natural frequency have been chosen as criteria to be minimized. The member cross sectional areas are the two design variables subject to side constraints. There are no other constraints. The truss is shown in Figure 9.

The numerical data are as follows: $\rho = 7850 \text{ kg/m}^3$, $L = 100 \text{ cm}$, Young's Modulus $E = 200 \text{ GPa}$, bar cross section area upper limit $A_v = 10.0 \text{ cm}^2$, bar cross section area lower limit $A_L = 2.0 \text{ cm}^2$, $m = \rho L (A_1 + \sqrt{2} A_2)/2$. The objectives are: criterion $c_1 = $ volume of the truss bars, $V$; criterion $c_2 = $ square of the lower natural frequency, $\omega_1^2$. 
The problem statement is

Minimize \( \{ V(A_1, A_2, A_3), \omega_1^2(A_1, A_2, A_3) \} \)

subject to \( A_L \leq A_i \leq A_U, i = 1, 2 \) \hspace{1cm} (21)

Using \( x \) as horizontal displacement and \( y \) as vertical displacement, the two degree of freedom vibration model has equations

\[
md^2 \frac{dx}{dt^2} - (xE/L)((A_1 + \cos^2(45)A_2)/\sqrt{2}) - (yE/L)(\cos^2(45)A_2/\sqrt{2}) = 0
\]

\[
md^2 \frac{dy}{dt^2} - (x + y)E A_2 \cos^2(45)/\sqrt{2} = 0
\]

The characteristic equation for the system is \( a\omega^4 + b\omega^2 + c = 0 \), where

\( a = m^2 \), \( b = (A_1 + A_2 \sqrt{2} \cos^2(45))(mE/L) \), \( c = A_1 A_2 E^2 \cos^2(45)/L^2 \sqrt{2} \). Then the criteria are \( c_1 = (-b - \sqrt{b^2 - 4ac})/2a \) and \( c_2 = \sqrt{2} LA_1 + LA_2 + 2 LA_3 \).
The attainable set is illustrated in Figure 10. The Pareto optimal points upon the non-convex boundary AB correspond to changes in the first design variable while the second is fixed at its lower bound. Only the two end points, A and B, can be found with the linear weighted criteria method by the sequential quadratic programming optimizer. Again, the exponential weighted criteria method can find every point on the non-convex curve. Large $m$ values are required to find the points upon the most concave part of the curve near endpoint A. With an $m$ value of three, points can be found from endpoint B to a fifth of the way to A. With an $m$ value of seven points can be found from B to two-thirds of the way to A. To find points along the entire boundary AB requires an $m$ value of fifty.

The WCP method is able to find all Pareto optimal points. For $m = 2$ only the end points are found, for $m = 5$ points from B to a fifth of the way to A are found, while to find points along the entire boundary AB required $m = 25$.

Example 3  A non-convex boundary in a constrained optimization

A simple example of Pareto optimal points on a non-convex boundary resulting from a non-convex constraint can be formulated as follows and is depicted in Figure 11.
Minimize \( c_1 = x_1, c_2 = x_2 \)

Subject to \( g_1(x_1, x_2) = 1 - x_1^2 - x_2^2 \leq 0 \)

\( g_2(x_1) = -x_1 \leq 0 \)

\( g_3(x_2) = -x_2 \leq 0 \) \hspace{1cm} (23)

Both criteria functions are convex with respect to the design variables but the constraint is non-convex. The Pareto set consists of all points on the arc that connects \( A \) and \( B \). Numerical minimization of the linear weighted objective function captures points \( A \) and \( B \). Because these are both local minima the algorithm converges to either one depending upon the starting point for the iterations. If the starting point is symmetric with respect to \( x_1 \) and \( x_2 \), the sequential quadratic programming method occasionally converges to points on the arc midway between \( A \) and \( B \), misled by the symmetry.

The exponential weighted criteria method with \( m = 1 \) can find only points \( A \) and \( B \). With \( m = 2 \), 40\% of the Pareto set can be found; with \( m = 5 \), 92\%; and with \( m = 16 \), the entire set can be found. The WCP method with \( m = 2 \) can find only points \( A \) and \( B \). With \( m = 3 \) the entire curve can be found.
WEIGHTED CRITERIA METHODS

When the problem was enlarged to three dimensions:

Minimize \( c_1 = x_1, c_2 = x_2, c_3 = x_3 \)

Subject to \( g_1(x_1, x_2, x_3) = 1 - x_1^2 - x_2^2 - x_3^2 \leq 0 \)
\( g_2(x_1) = -x_1 \leq 0 \)
\( g_3(x_2) = -x_2 \leq 0 \)
\( g_4(x_3) = -x_3 \leq 0 \) (24)

the results obtained were approximately the same as for the two dimensional problem.

CONCLUDING REMARKS

There may be many instances in which the decision maker cannot be assured that an attainable region is convex. Therefore it appears reasonable to use a method capable of performing well with attainable regions that are not convex. The two weighted criteria methods analyzed in this article are able to find all Pareto optimal points regardless of attainable set convexity, provided an appropriate control parameter \( m \) is used.

Acknowledgements

This research was partially supported by a graduate fellowship grant from General Motors Corporation. This support is gratefully acknowledged. The authors wish to thank Professor Freerk Lootsma for his many helpful suggestions.

References


