A QUASI-MONTE CARLO METHOD FOR MULTICRITERIA DESIGN OPTIMIZATION

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(Received 7 April 1995)

The Quasi-Random Weighted Criteria method is proposed for multicriteria design optimization. This quasi-Monte Carlo method has increased computational efficiency and is particularly suitable for exploring alternative design configurations, or alternative strategies in controller design. A quasi-random sequence of test points generates a set of candidate solutions representative of the range of available solutions for each design alternative. The method can be used recursively to produce more detailed Pareto surface descriptions near selected points. The new concept of the meta Pareto set is introduced to represent the best compromise solutions for all design alternatives under consideration. A linear controller design is included as an illustrative application.

Keywords: Multiobjective optimization; optimal design; controller design

INTRODUCTION

Multicriteria (or multiobjective) optimization (MCO) is the search for a solution that best manages tradeoffs between noncommensurable criteria \[1, 2\]. The minimization formulation of an MCO problem is

\begin{equation}
\begin{aligned}
\text{minimize } c(x, b) \\
x, \\
\text{subject to } g(x, b) \leq 0 \\
h(x, b) = 0
\end{aligned}
\end{equation}

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where \( \mathbf{c} \) is the vector of criteria. The design variables \( \mathbf{x} \) that satisfy constraints \( \mathbf{g}(\mathbf{x}, \mathbf{b}) \) and \( \mathbf{h}(\mathbf{x}, \mathbf{b}) \) constitute the feasible set, \( \mathbf{F} \). The corresponding values for \( \mathbf{c}(\mathbf{x}, \mathbf{b}) \) constitute the attainable set, \( \mathbf{A} \). MCO methods generally require a series of optimizations with a scalar substitute problem generally defined as

\[
\begin{align*}
\text{minimize } & f(\mathbf{c}, \mathbf{M}) \\
\text{subject to } & \mathbf{g}(\mathbf{x}, \mathbf{b}) \leq 0 \\
& \mathbf{h}(\mathbf{x}, \mathbf{b}) = 0
\end{align*}
\]

with \( f \) a scalar function, and \( \mathbf{M} \) a set of preference parameters (weights or other quantities) adjusted by the decision maker to tune the objective function so that it matches the decision maker's preferences. An MCO method should provide a systematic way for adjusting the scalarization parameters of substitute problems.

Methods for problems with large numbers of criteria can carry large computational costs. In some the decision maker's selection of preference parameters in the scalar substitute problem is no longer the focus of the solution effort. Rather, these methods attempt to provide many candidate solutions from which the decision maker can pick the most preferred, and \textit{a priori} selection of preference parameters is eliminated altogether. The decision maker can instead focus directly upon the available design solutions.

Such methods should be straightforward and transparent in their operation to facilitate their use, particularly since intuitive understanding of tradeoffs for MCO problems with more than three criteria is generally very difficult. The percentage of a multidimensional criteria region that is feasible in a typical real world multicriteria design problem has been estimated as less than 0.1% [3]. This differs from the impression obtained from problems with two or three criteria. Organization and interpretation of data is a challenge. Additionally, computational efficiency becomes an important aspect of a method's performance, for the computational burden can become impractically large. An estimate has been made that with large solution samples the Pareto set, on
average, receives a number of points equal to the logarithm of the sample size [4]. To obtain fourteen Pareto optimal points a sample of one million attainable set points is required.

Finally, a decision maker often faces design situations with a number of alternatives or design strategies. Examples of alternative design strategies are different control algorithms for controller design, alternative sites for a civil engineering site planning project, different hardware configurations in mechanical design, and different investment strategies. The method presented below generates a solution set that is representative of the range of optimal solutions for each strategy, and therefore it is useful in quantitative comparison between strategies.

Three MCO approaches directly related to the new method are now briefly reviewed.

**Meisel’s Method. [5]**

Meisel [5] proposed the use of a random input to direct the selection of weights for a weighted criteria formulation in repeated scalar substitute optimizations. More particularly

\[ w_{i}^{l}(k) = r w_{i}^{l-1} \text{ for } l = 1 \text{ to } L \text{ optimizations} \quad (3) \]

where \( w_{i}^{l} \) is the weight for the \( i \)th criterion on the \( l \)th optimization, and \( r \) is a non-negative random variable with mean value one. The objective function for the \( l \)th optimization is

\[ f = w_{1}^{l}c_{1} + w_{2}^{l}c_{2} + \ldots + w_{I}^{l}c_{I} \quad (4) \]

Meisel does not specify the distribution of the random variable. With repeated optimizations Meisel’s method can generate large numbers of points distributed over the attainable set. Because the weights are randomly distributed the method is not efficient. A very large number of scalar optimization calculations is required to guarantee that the solution set is thoroughly distributed across the attainable set, because the first few random numbers may generate outlier weights.
Meisel’s method does not capture Pareto optimal points corresponding to an even distribution across the ranges of possible criteria weight ratios, because the weight ratios determined from Eq. (3) will not be evenly distributed. For example, the same weight ratios will occur each time all weights are equal, whether with values of 0.1, 0.7, or 1.8. Yet many ratios of weights will occur quite infrequently, e.g. if the random number distribution spans 0.5 to 1.5, a weight ratio greater than 3.0 would never occur. This unevenness has the effect of increasing the representation of points determined with equal weight with respect to points in which one or more criteria receive greater relative weighting. In some applications this bias may be allowable or even desirable, but it would impose an undesirable bias in other applications.

**Osyczka’s Method. [6]**

Osyczka [6] suggests the use of a Monte Carlo method to pick design variables over their estimated ranges. Criteria values are calculated for each set of selected design variables. This process generates a set of feasible points. Most points will not be Pareto optimal and therefore the Pareto set must be extracted from the set. Many calculations of the criteria values will generally be required, and even then there is no certainty that the set adequately represents the attainable Pareto set.

**Statnikov’s Method. [3, 7–9]**

Rather than utilizing randomness to pick design variables Statnikov [3, 7–9] suggested a search strategy to generate a representative set of feasible points. The decision maker designates minimum acceptable target values for each objective. Then, using quasi-random sequences (described further below) a uniformly distributed scattering of feasible trial points is generated. Statnikov’s method uses these sequences to deliver a uniform and efficient distribution for the design variables.

Because Statnikov’s method is similar to Osyczka’s in that it is based upon the generation of a large set of feasible points from which Pareto optimal points must be extracted, it too will require many calculations of the criteria values. Statnikov reports that as many as 20,000 feasible solutions have been generated in a design study. Stadler and Dauer [10]
note that there is presently no assurance, other than practical experience, that this approximation converges in some fashion to the set of Pareto optimal solutions for a given multicriteria optimization problem.

Statnikov reported that his method was widely used in the Soviet Union and that he has participated in its application to literally hundreds of problems in every sphere of mechanical engineering [3, 7–9]. This wide application is testimony to the usefulness of methods for design problems with many criteria.

An improved method that overcomes some of the difficulties of the methods just described is introduced next.

THE QUASI-RANDOM WEIGHTED CRITERIA METHOD

The Quasi-Random Weighted Criteria (QRWC) Method [11] solves a series of scalar substitute problems with systematically adjusted scalar objective weights. This series of scalar optimizations is termed a pattern of optimizations, or a pattern. Multiple patterns may be undertaken in a design study of alternative strategies. Each pattern provides an assessment of one design strategy. Multiple patterns provide comparisons among design strategies.

The method utilizes quasi-random sequences to generate the weights. Quasi-random sequences cover a hypervolume evenly and efficiently, while at the same time maintaining random-like relations between the dimensional components. Despite the misleading name these sequences are deterministically generated and were developed to convert the statistical integration of Monte Carlo methods to the numerical integration of quasi-Monte Carlo methods. They improve the efficiency of Monte Carlo methods by eliminating the reliance upon random or pseudo-random sequences.

The QRWC method may be described in pseudo-code as follows:
For \( l = 1 \) to \( L \)
Get weights \( w_l \) from quasi-random series
Minimize \( (w_l)^T c \) subject to constraints
Tabulate results
Next \( l \)
The scalar objective utilizes a set of relative weights to compute a Pareto optimal point. A distribution over the ranges of all criteria weight ratios is produced with the quasi-random sequences. The distribution is chosen
to provide optimal solutions that can be considered representative of the potential combinations of criteria weights. A representative solution set could be defined in a number of ways. One definition would be that the weights, $w_i$, are uniformly distributed. In the QRWC method weight ratios are used as a better reflection of the preference structure. As will be shown below, the end result is that Pareto points are computed in an evenly distributed manner over the entire attainable set.

The method can be used recursively. After a set of candidate solutions is generated, a subset is selected by the decision maker and a new set of candidate solutions is generated within the neighborhood of the selection. The process can be repeated with increasingly smaller neighborhoods until the decision maker selects a final solution.

**Selection of Representative Weight Ratios.**

Consider the two criteria illustration in Figure 1. If weights $w_1$ and $w_2$ were independently determined quasi-randomly with values between 0 and 1, the weight vectors $(w_1, w_2)$ would be uniformly distributed over the square area $w_1w_2$. Yet this would heavily bias weight ratios near one. Each ray from the origin will intersect only points with the same relative weighting. Rays of different directions would intersect different total

![Figure 1](image-url)  
**FIGURE 1** Uniform distribution across a two dimensional region.
numbers of points within the square area. For example, the ray that consists of points of equal weighting (directed 45 degrees above the horizontal axis) would tend to intersect $\sqrt{2}/\sqrt{1.25}$ times as many points as a ray with 1-to-2 weighting. This result indicates that if the weights were picked in this manner the relative weighting would tend to more often to 1-to-1 than 1-to-2.

In the QRWC method the quasi-random sequences are projected into an (I-1) dimensional simplex when I is the number of criteria. In Figure 1 this consists of the convex combinations contained on the simplex line with vertices at (1,0) and (0,1). This reduces the bias towards weight ratios of one, but does not eliminate it.

An alternative representative set would have a uniform distribution over the weight ratios between one and infinity, and symmetric to the weight ratios between zero and one. This would have no bias. Yet it is unlikely that a decision maker would be as interested in extremely large weight ratios near one. The bias resulting from use of the simplex distribution is considered reasonable.

Using quasi-random vector sequences for the weights also provides an efficient pattern of scalar substitute problem optimizations for exploring the attainable set. As described below, the low discrepancy of quasi-random sequences enables them to cover efficiently the range of possible weighting combinations. Because of their relative spacing, each point represents a maximal range of weight ratios, and therefore provides a maximal amount of new information. At the same time the random relations between vector components prevents criteria interactions from influencing the results.

**Generation of the Quasi-Random Sequence**

In the QRWC Method, (I-1) components of the weight vector will be calculated as independent quasi-random numbers from the Hammersley point set. This set is a finite sequence of points created to cover a hypercube uniformly and with low discrepancy. Discrepancy is a measure of the efficiency of a series in covering a volume. A low discrepancy sequence is one that is distributed such that each point is near a large amount of hypervolume that is not near other points. The following definition is a hybrid of existing definitions of discrepancy.

**DEFINITION 1** Consider an I dimensional region $B$ inside the I-dimensional unit hypercube $K$. The hypervolume of $B$ is $V_B$. Let
be a sequence of points in $K$. The number of points inside $B$ will be denoted $\phi(B)$, i.e. $\phi(B)$ is the number of elements in the set \{\(\xi_i \mid \xi_i \in B, \ i = 1, \ldots, L\}\}. Then the local discrepancy of the sequence $S$ in the hypercube $K$ can be measured by the difference

$$ D(B) = \frac{\phi(B)}{L} - V_B $$

which gives the difference between the relative frequency and the probability that a point falls into $B$. 

From the definition of local discrepancy a number of specialized definitions of global discrepancy can be made. Following the exposition by Niederreiter [12] they take the general form

$$ D_f(B;S) = \sup(T \cap B) |D(B)| $$

where $T$ is a nonempty family of subsets of the unit hypercube $K$, and $S$ is the point set under consideration.

Because the Hammersley point set was devised to cover a hypercube, not a simplex, a projection into a simplex must be made. After the first (I-1) weights, $\xi_i$, are generated, $w_i$ is set to $\xi_i$ for $i = 1$ to (I-1), and $\sum_{i=1}^{I-1} w_i = 1$ is imposed. If the sum of the (I-1) components is greater than one, this vector is rejected. If not, $w_i = 1 - \sum_{i=1}^{I-1} w_i$.

**Theorem 1** The uniform and low discrepancy distribution that quasi-random numbers provide for an (I-1) dimensional hypercube can be converted to a uniform distribution with lower discrepancy across a simplex. This conversion is made by calculating (I-1) components, then imposing $\sum_{i=1}^{I-1} w_i = 1$ to determine the $i$th component. If the sum of the (I-1) components is greater than one, this vector is rejected.

**Proof** There are two steps. First, a polygonal fraction of the (I-1) hypercube defined by $0 \leq \xi_i \leq 1$ is discarded, leaving polygon $W$. (The volume in which $\sum_{i=1}^{I-1} w_i \geq 1$ is discarded). Because $\xi_i$ are uniformly distributed with low discrepancy over the hypercube, they are also uniformly distributed with low discrepancy over the selected hypervolume. Specifically,

$$ D_f(S)_{\text{over } W} \leq D_f(S)_{\text{over } K} $$

(7)
where $D_I(S)^\text{over } w$ is the discrepancy of an $I$ member point set, $S$, over hypervolume $W$, because the supremum is being formed for the same sequence but over a smaller set of convex spaces.

In the second step the first $(I-1)$ terms of the vector $w$ are set equal to the $(I-1)$ terms of the vector $\xi$. Then $w_I$ is determined as $1 - \sum_{i=1}^{I-1} w_i$. This mapping projects the point from the $(I-1)$ dimensional polygon onto an $(I-1)$ dimensional simplex. The projection is a linear operation, because the simplex and the polygon are of the same dimension, and the projection is bijective (i.e., both one-to-one and onto). It is one-to-one because the $(I-1)$ components in $W$ uniquely determine $w_I$. It is onto because if the $I$ components of a point in the simplex, $W$, are specified then the first $(I-1)$ components of this point determine a unique point $\xi$ in $W$.

Because the second step is a linear projection the local discrepancy will be unchanged by it. If the projection scales dimension $i$ by factor $f_i$ for every $i$ from 1 to $(I-1)$, then for the region $U$ of hypervolume $V_U$ within $W$ contains one of the $L$ points in sequence $S$ the local discrepancy is

$$D(S) = \frac{1}{L} - V_U. \quad (8)$$

The projection of region $U$ on the simplex will still contain $1/L$ of the points in the sequence. The hypervolume of the projection of $W$ will be $V_U f_1 f_2 \ldots f_I$. The hypervolume of the entire simplex will be $f_1 f_2 \ldots f_I$. Therefore the ratio of the hypervolume of region $U$ to the hypervolume of the simplex will be $V_U$ as before (the second term in the expression for discrepancy).

In 1990 Deak [13] stated that there are no known results concerning the construction of point sequences over a hypersphere or other domains. That same year a low discrepancy point set for simplices, hyperspheres and other domains was published by Wang and Fang [14]. They transform a uniformly distributed series with low discrepancy for hypercubes, $S: (\xi^1, \ldots, \xi^L)$, to series $W: (w^1, \ldots, w^L)$ with uniform distribution over other regions. Quasi-random sequences are frequently used for approximating an integral. The discrepancy of $W$ is such that if

$$\left| \int_K f(x) dx - \frac{1}{L} \sum_{i=1}^L f(\xi^i) \right| \leq V(f) D(S) \quad (9)$$
where $V(f)$ is the total variation in the sense of Hardy and Krause, and $D(S)$ is a measure of discrepancy, then

$$\left| \int_R f(x)dx - \frac{1}{L} \sum_{l=1}^{L} f(\alpha_l) \right| \leq V(f)D(W)V_B$$  \hspace{1cm} (10)

where $V_B$ is the hypervolume of the simplex. For simplicies of dimension (I-1) they derive the formula for series $w^l = (w_1^l, \ldots, w_I^l)$:

$$w_i^l = \prod_{u=1}^{i} \xi_u \hspace{1cm} (\xi_u)^{1/(l-u+1)}, \hspace{1cm} l = 1, \ldots, L; \hspace{1cm} i = 1, \ldots I.$$  \hspace{1cm} (11)

This procedure is an alternative to the method used in this paper. The error limit in the approximation to mean function values is larger with this second method. By Eq. (9) the error has an upper limit of $V(f)D(W)V_B$. By Theorem 1 the discrepancy for the proposed method will be less than $V(f)D(W)$.

Niederreiter [12] in his recent book on quasi-Monte Carlo methods states that it is widely believed that the Hammersley point set with pairwise relatively prime bases attains the discrepancy of smallest possible magnitude. He notes that this result has not yet been proven. No two numbers in a pairwise relatively prime set share a multiplicative factor other than the number one.

Hammersley points sets were used, but without pairwise relatively prime bases, in the demonstration below. Instead, the bases were integers selected sequentially, which is expected to increase the discrepancies by a small amount.

The Hammersley and the Shifted Hammersley Points Sets.

There are a number of quasi-random series and points sets. The point set used in this work is termed the shifted Hammersley sequence, originally suggested by van der Corput [15] and developed by Halton [16] and Hammersley [15].

Any positive integer $l$ can be expressed in any integer base $b$ as

$$l = a_m + a_{m-1}b + \ldots + a_1b^{m-1}.$$  \hspace{1cm} (12)
DEFINITION 2  [Hammersley and Handscomb 17] For any integer \( b \geq 2 \), the radical-inverse function \( F_b(l) \) is defined on these integers as

\[
F_b(l) = a_m b^{-1} + a_{m-1} b^{-2} + \ldots + a_1 b^{-m}
\]  

(13)

with \( l \) expressed in base \( b \) as in Eq. (12). The radical inverse function reflects the numbers \( l \) about its “decimal” point.

The quasi-random Hammersley point set is:

\[
w^l = \left( l/L, F_{b_1}(l), F_{b_2}(l), \ldots, F_{b_{[L-1]}}(l) \right)
\]

(14)

where \( L \) is the total number of points in the series, \( b_i \) are the integer bases, and \( I \) is the number of components in the weight vector, \( w \).

Shifting the first component is advantageous. Wozniakowski [18] shows that shifting the first component to \( (l + t)/L \), with \( 0 \leq l + t \leq L \) reduces the average case error in numerical integration for one class of functions. Wozniakowski did not specify a value for \( t \). This parameter was set to \(-0.5\) in the demonstration in this paper. That value makes the first component of the series symmetric over the segment \((0,1)\). The unshifted Hammersley sequences approach this symmetry asymptotically as \( L \) increases, but because in the QRWC method the point sets used are often very small, establishing this symmetry was judged desirable. The shifted Hammersley sequence selected is

\[
w^l = \left( (1 - 0.5)/L, F_1(l), F_2(l), \ldots, F_{I-1}(l) \right)
\]

(15)

This concludes the presentation of the theoretical justification for the QRWC method.

THE META PARETO SET

The new term meta Pareto set is proposed for MCO studies. It is useful in design studies which compare multiple strategies.

DEFINITION 3  Given Pareto sets \( P_1, P_2, P_3, \ldots, P_p \) for \( p \) strategies for a design problem, the meta Pareto set consists of points within the union of these sets, \( P = P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_p \) that are Pareto optimal with respect to set \( P \). The vector \( c^0 \) is meta Pareto optimal if and only if \( c_i^0 \leq c_i \) for all \( i \) and \( c_i^0 \) for at least one \( i \), for all \( c \in P \).
Each design strategy will possess a Pareto set. From these Pareto sets a higher level of relatively dominant solutions can be found, the meta Pareto set. The union of the Pareto sets forms a higher level attainable set and its own Pareto set is the meta Pareto set. This idea is illustrated with a two criteria design problem in Figure 2.

The meta Pareto set represents the best performance available from all strategies under consideration. When the configurations are control algorithms, the meta Pareto set can suggest adaptive control strategies. For example, from Figure 2, if criterion $c_1$ is important under one set of stochastic conditions, while criterion $c_2$ is important under a different set, and if the transition from one set to the other is predictable, the control algorithm would be changed from one to the other as the conditions change.

**A DEMONSTRATION: DESIGN OF A LINEAR CONTROLLER**

A closed loop controller is to be designed. Six criteria are important.

1. *Rise Time of a Step Input Response*. Rise time is quantified as the time to first attain the steady state response value.
2. *Overshoot in a Unit Step Input Response*. Overshoot is the difference between the response's maximum value and the steady state value.
3. *Relative Stability*. Stability is quantified as the distance from the imaginary axis to the nearest root of the characteristic equation of the closed loop transfer function.

![Diagram](image)

**FIGURE 2** The meta Pareto set.
4. **Integrated Square Error (ISE)**. This is the squared error in a step input response, integrated from start to steady state.

5. **Steady State Error to a Ramp Input**. The algorithms have an error at steady state in their responses to ramp input.

6. **Sensitivity of ISE to Changes in the Plant**. The change in ISE due to change in the plant transfer function is used as a measure of controller robustness.

All criteria are to be minimized with the exception of relative stability, for which larger values are preferred to smaller. The controllers are constrained to operate with overdamped response. The plant to be controlled is characterized as

\[
\frac{d^2 y}{dt^2} + \frac{dy}{dt} = u
\]  

(16)

where \( y \) is plant output and \( u \) is plant input. Two algorithms are considered: Derivative Control (Fig. 3) and Velocity or Tachometer Control (Fig. 4).

Each of the control schemes has two design variables: a gain, \( K \), and a time constant, \( T \). These variables are assumed positive and the gain cannot be larger than 50.

\[
g_1: K > 0.0 \quad g_2: T > 0.0 \quad g_3: K \leq 50.
\]  

(17)

An initial criteria specification is selected for scaling:

- Ramp SS Error \((c_1)\): Less than 1.0
- Overshoot \((c_2)\): Less than 0.4
- ISE \((c_3)\): Less than 0.5
- Error Sensitivity \((c_4)\): \(-0.5\) to 0.5, Smaller Magnitudes Preferred

![FIGURE 3 Derivative control.](image-url)
Rise Time \( (c_4) \): Less than 0.5 second  
Relative Stability \( (c_6) \): Greater than 0.8.

The criteria functions were scaled in the objective function by the above specifications resulting in a weighted criteria objective function of

\[
f = w_1(c_1) + w_2(2.5c_2) + w_3(2c_3) \\
+ w_4(2.0|c_4|) + w_5(2c_5) + w_6((1.6 - c_6) \cdot 1.25)
\]  \( \text{(18)} \)

**Pattern Results**

The sequential quadratic algorithm NLPQL [19] was used. The number of optimizations was initially chosen to be six, based upon experience with the algorithms. If too few points are selected the solution set might not adequately represent the algorithms for comparison purposes. If the number is large the presentation of results is more difficult. The results are presented in Table I.

The derivative control produced a smaller objective function value for the second, fourth and fifth optimizations. In all optimizations the tachometer control produced less overshoot, but larger ISE. Generally it had lower sensitivity. These results are seen more clearly in the summaries of Table II and in Figures 5 and 6.

Figures 5 and 6 are two-dimensional projections of the three critical criteria in this comparison. To examine these three criteria together, a three dimensional surface approximation was made using a second order polynomial. The two sets of six points shown in these figures were
TABLE I  Solution points from a six optimization pattern

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Ramp Error</th>
<th>Overshoot</th>
<th>ISE</th>
<th>Sensitivity</th>
<th>Rise Time</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0200</td>
<td>0.1572</td>
<td>0.04866</td>
<td>−0.003090</td>
<td>0.8660</td>
<td>6.402</td>
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<tr>
<td>2</td>
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<td>0.1685</td>
<td>0.05100</td>
<td>−0.003100</td>
<td>0.4491</td>
<td>5.000</td>
</tr>
<tr>
<td>3</td>
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<td>0.1685</td>
<td>0.05100</td>
<td>−0.003100</td>
<td>0.4491</td>
<td>5.000</td>
</tr>
<tr>
<td>4</td>
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<td>0.3643</td>
<td>0.1018</td>
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<td>0.2714</td>
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<tr>
<td>5</td>
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<td>5.000</td>
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<tr>
<td>6</td>
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<td>0.05069</td>
<td>−0.003161</td>
<td>0.4658</td>
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Tachometer

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Ramp Error</th>
<th>Overshoot</th>
<th>ISE</th>
<th>Sensitivity</th>
<th>Rise Time</th>
<th>Stability</th>
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TABLE II  Pattern summaries

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<th>Objective</th>
<th>Ramp Error</th>
<th>Overshoot</th>
<th>ISE</th>
<th>Sensitivity</th>
<th>Rise Time</th>
<th>Stability</th>
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<tr>
<th>Deriv.</th>
<th>Maximum Values</th>
<th>Mean Values</th>
<th>Minimum Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2902</td>
<td>0.0216</td>
<td>0.3643</td>
<td>−0.09305</td>
</tr>
<tr>
<td>0.3264</td>
<td>0.1928</td>
<td>0.3487</td>
<td>−0.08394</td>
</tr>
<tr>
<td>−0.0353</td>
<td>0.0200</td>
<td>0.1572</td>
<td>−1.390</td>
</tr>
<tr>
<td>−0.0163</td>
<td>0.04866</td>
<td>0.1482</td>
<td>−0.0147</td>
</tr>
</tbody>
</table>

used. For the derivative control algorithm results the surface was described as

\[ c_4 = -0.00427 + 0.346c_2 - 2.14c_2^2 - 1.11c_3 + 6.68c_2c_3 + 1.14c_3^2 \] (19)

and for the tachometer control algorithm results

\[ c_4 = 0.00770 - 0.0167c_2 - 0.0943c_2^2 - 0.0236c_3 \]
\[ + 0.0703c_2c_3 + 0.0978c_3^2 \] (20)

These three dimensional surfaces are plotted in Figures 7 and 8. These approximations extrapolate over the plotted region, and there is no
assurance that the entirety of either surface is actually attainable. A comparison of the two plots does indicate relative performances. The spider plot format may be the best for comparing multiple criteria. Each criterion is plotted on a separate axis. Figure 9 is a spider
plot of the normalized mean criteria values. The differences are exaggerated by the normalization which was used to simplify the graph.

For all runs, with both controllers, constraint $g_3$ was active, because four of the criteria (ISE, sensitivity, stability and ramp error) monotonically improve with increases in the first variable. Rise time initially improves, but degrades for large values; and overshoot initially degrades, but improves for large values. The ramp error criterion is a function of only the first variable and therefore was fixed in value by this active constraint. The number of optimizations was increased to fifty to determine if six was producing a Pareto set too small to be representative. This provided an increased range of solution points, but with the same relative conclusions, as reflected in the summaries of Table III.

The derivative controller appears generally preferable, though the smallest objective value found belongs to the tachometer controller. The
decision between the two depends upon the relative importance of the criteria. A shortcoming of the derivative controller is the large overshoot. To investigate whether this can be avoided, the six pattern evaluation was repeated, with the weighting of the rise time term increased by a factor of ten. This scaling moves solutions into a region of the Pareto set that has lower rise time criterion values. Doing so will reveal the compromises to the other criteria that must be made to realize lower rise times. The summaries that result are shown in Table IV. This table should be compared with Table II.

TABLE III  Optimization summaries for derivative and tachometer control for 50 optimizations

<table>
<thead>
<tr>
<th></th>
<th>Objective</th>
<th>Ramp Error</th>
<th>Overshoot</th>
<th>ISE</th>
<th>Sensitivity</th>
<th>Rise Time</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>0.3167</td>
<td>0.02000</td>
<td>0.2931</td>
<td>0.08090</td>
<td>-0.002126</td>
<td>10.89</td>
<td>7.071</td>
</tr>
<tr>
<td>Mean</td>
<td>0.3562</td>
<td>0.02000</td>
<td>0.2351</td>
<td>0.2334</td>
<td>0.007406</td>
<td>2.921</td>
<td>6.999</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.8724</td>
<td>0.02000</td>
<td>0.1637</td>
<td>0.05117</td>
<td>-0.003360</td>
<td>1.172</td>
<td>6.041</td>
</tr>
<tr>
<td>Tach.</td>
<td>-0.8578</td>
<td>0.02000</td>
<td>0.02851</td>
<td>0.1868</td>
<td>0.005708</td>
<td>0.9406</td>
<td>5.947</td>
</tr>
<tr>
<td></td>
<td>-2.675</td>
<td>0.02000</td>
<td>0.1252</td>
<td>0.04808</td>
<td>-0.0097</td>
<td>0.3000</td>
<td>3.152</td>
</tr>
<tr>
<td>Tach.</td>
<td>-3.058</td>
<td>0.02000</td>
<td>0.0000</td>
<td>0.14144</td>
<td>-0.0043</td>
<td>0.3118</td>
<td>2.959</td>
</tr>
</tbody>
</table>
TABLE IV  Optimization summaries for derivative control for 6 optimizations with weighted overshoot

<table>
<thead>
<tr>
<th>Ramp Error</th>
<th>Overshoot</th>
<th>ISE</th>
<th>Sensitivity</th>
<th>Rise Time</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1574</td>
<td>0.2358</td>
<td>0.1347</td>
<td>0.005274</td>
<td>1.733</td>
<td>6.516</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum Values</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.04819</td>
<td>0.1345</td>
<td>0.06737</td>
<td>-0.001661</td>
<td>1.013</td>
<td>4.585</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mean Values</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0200</td>
<td>0.0704</td>
<td>0.04856</td>
<td>-0.0060</td>
<td>0.3399</td>
<td>2.1477</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Minimum Values</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The overshoot remains much larger than the tachometer control overshoot, while this reduction was obtained at the cost of a much increased rise time and an increased ramp error. The larger ramp error values indicate that the third constraint is no longer always active. The optimum first design variable value ranged from 50 to 6.35. These results are depicted in a plot of normalized mean values, Figure 10. The mean values have been normalized by the derivative control mean values in Table II. The third bar in each set represents the mean values from Table IV. From the figure it is clear that overshoot was not reduced much relative to increases in ramp error.

FIGURE 10  Mean criteria values normalized to initial derivative control values.
Selection of Compromise Solution

The final selection of a control scheme will depend upon the preferences of the decision maker. The derivative controller generally performs better, but these analyses indicate its relative weaknesses in overshoot and sensitivity. Additional applications of the method may be used to understand more thoroughly the strengths and weaknesses of the two algorithms; the work just described demonstrates a typical approach. Assumptions will now be made to demonstrate a typical approach to selecting a final solution. It is assumed that the derivative controller is preferred due to its low error. Rise time is a concern. From the six optimizations made with increased rise time weighting a solution is selected:

\[
x = \{50.0, 0.134\} \quad c = \{0.020, 0.236, 0.0664, -0.006, 0.334, 3.84\} \quad (21)
\]

\[
w = \{0.294, 0.0234, 0.263, 0.0350, 0.384, 0.000186\}. \quad (22)
\]

The factor of ten increase to the weighting of overshoot is in addition to these weights. Another pattern of optimizations was conducted, focused upon the neighborhood of this solution point. The weights are centered upon the Eq. (23) weight vector. The range of the quasi-random sequence was reduced to one-sixth of its previous range, to center it about this solution which had been one of six.

The results are presented in Table V. These six candidate points all lie close to the previously selected solution. One of these points would now be selected, as a final solution, or as a focus for a more finely focused QRWC method pattern of optimizations.

<table>
<thead>
<tr>
<th>TABLE V</th>
<th>Candidate solutions from a focused QRWC method pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ramp Error</td>
</tr>
<tr>
<td>1</td>
<td>0.0200</td>
</tr>
<tr>
<td>2</td>
<td>0.0200</td>
</tr>
<tr>
<td>3</td>
<td>0.0200</td>
</tr>
<tr>
<td>4</td>
<td>0.0200</td>
</tr>
<tr>
<td>5</td>
<td>0.0200</td>
</tr>
<tr>
<td>6</td>
<td>0.0200</td>
</tr>
</tbody>
</table>
CONCLUDING REMARKS

The QRWC method is simple to understand and apply. There is a good theoretical foundation to justify the expectation of relative high efficiency, based on the low discrepancy of the shifted Hammersley sequence and the fact that only Pareto points are computed, so no computations are wasted. The method is suitable for design situations in which the decision maker has no a priori strategy for establishing bounds or weights for the performance criteria that need to be balanced. This is particularly true for problems with large numbers of criteria where physical intuition on tradeoffs is defied by the dimensionality of the problem. Once an understanding of the overall performance of a design strategy is attained the method can be used iteratively with increasingly focused patterns of optimizations to select a particular final solution.

Acknowledgment

This research was partially supported by a graduate fellowship grant from General Motors Corporation. This support is gratefully acknowledged. The authors wish to thank Professor Freerk Lootsma for his many helpful suggestions.

References


