

## Detection of degenerate normal vectors on parametric surfaces: Tangent cone approach

Deok-Soo Kim <sup>a,\*</sup>, Panos Y. Papalambros <sup>b</sup>

<sup>a</sup> *Department of Industrial Engineering, Hanyang University, Haengdang-Dong, Sungdong-Ku, Seoul, South Korea*

<sup>b</sup> *Department of Mechanical Engineering & Applied Mechanics, The University of Michigan, Ann Arbor, MI, USA*

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### Abstract

Normal vectors of a surface are essential in various geometry-related applications. Since a normal vector at a point on a parametric surface is computed by a cross product of two tangent vectors at the point, a regular parametric surface may possess a degenerate normal vector at a point when tangent vectors of the surface are linearly dependent at the point. This paper describes an algorithm which tests if a regular parametric surface does not possess such a degenerate normal vector.

*Keywords:* Degenerate normal vector; Tangent; Normal; Cone; Map; Bézier surface

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### 1. Introduction

Normal vectors of a surface are essential in many geometry-related applications such as surface rendering and surface machining. Even though normal vectors always exist at points on a regular surface, normal vectors on a parametric surface may degenerate since a normal vector at a point on a parametric surface is computed as a cross product of two tangent vectors of the surface. A degenerate normal vector occurs at a point on a parametric surface when the partials at the point are linearly dependent. When a surface has points with degenerate normal vectors, application programs such as machining software may face difficulties (Kim, 1990; Yamaguchi, 1988). Therefore, when a degenerate normal vector exists on a parametric surface, care should be exercised for the applications.

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\* Corresponding author.

This paper presents an algorithm which uses tangent cones to make sure that there does not exist a point with degenerate normal vector on a regular parametric surface. It has been known that tangent cones of a parametric surface can be computed from hodographs (Sederberg and Meyers, 1988; Kim, 1990; Kim et al., 1995). There are two tangent cones on a parametric surface, one for each parameter. Since a degenerate normal vector occurs when the partials are linearly dependent, a parametric surface cannot contain a degenerate normal vector if any pair of tangent vectors from each tangent cone is not linearly dependent. Section 2 describes the conditions for a degenerate normal vector to occur on a parametric surface. Section 3 provides the algorithm via tangent cones and maps. Then, this paper concludes.

## 2. Degenerate normal vectors

A normal vector of a point on a parametric surface  $B(u, v)$  at  $u_0$  and  $v_0$  is defined as a cross-product of two partial derivatives such as

$$\mathbf{n}(u_0, v_0) = \frac{\partial}{\partial u} B(u_0, v_0) \times \frac{\partial}{\partial v} B(u_0, v_0) \quad (1)$$

When the partials,  $(\partial/\partial u)B(u_0, v_0)$  and  $(\partial/\partial v)B(u_0, v_0)$ , are linearly dependent, a normal vector degenerates at  $u_0$  and  $v_0$ , and the normal vector at the point should not be computed by Eq. (1). A normal vector degenerates when two partials are in one of the following cases:

$$(1) \quad \frac{\partial}{\partial u} B(u, v) = \mathbf{0} \quad \text{and/or} \quad \frac{\partial}{\partial v} B(u, v) = \mathbf{0}. \quad (2)$$

This case occurs when one of two partials or both become(s) null vector(s). This happens when the control points of a boundary curve of a surface collapse into a point. When this happens, the boundary curve itself degenerates into a point and the partials on the curve are undefined. (Farin, 1990) discusses the details of this degenerate case.

$$(2) \quad \frac{\partial}{\partial u} B(u, v) = \lambda \frac{\partial}{\partial v} B(u, v), \quad \lambda \neq 0. \quad (3)$$

This case occurs when two partials are defined, but are collinear. Fig. 1 illustrates examples when this case occurs at a corner of a surface. Figs. 1a and 1b depict cases when  $\lambda > 0$  and  $\lambda < 0$ , respectively. Since a derivative at an extreme point on a curve is defined by the extreme edge of a control chain of the curve, Eq. (3) is satisfied when three control points  $b_{23}$ ,  $b_{32}$  and  $b_{33}$  are collinear as shown in the figure. Therefore, the normal vector of point  $B(1, 1)$  should not be computed by a cross-product of two partials at the point.

Previous examples illustrate cases when a degenerate normal vector occurs at a boundary of a surface, and the detection of these cases can be easily done by checking the boundary control points of a surface.

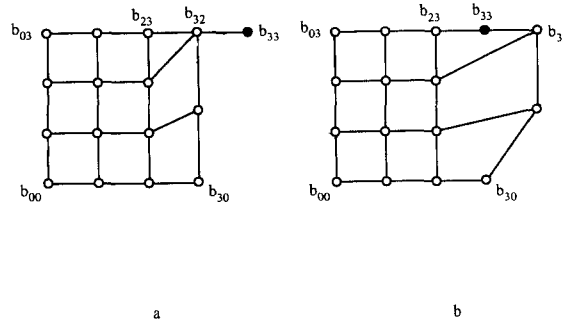


Fig. 1. Control polygons of surfaces with degenerate normal vectors at corners. (a) Degenerate normal vector with  $\lambda > 0$  at  $b_{33}$ . (b) Degenerate normal vector with  $\lambda < 0$  at  $b_{33}$ .

However, a normal vector does not necessarily degenerate at a boundary of a surface, but at any point on a surface when either of Eqs. (2) or (3) is satisfied. Fig. 2 shows an example of a planar bi-linear surface with degenerate normal vectors interior to the surface. Assuming that the surface lies on the  $XY$ -plane, a normal vector at  $u = v = 0$  is toward a positive  $Z$ -direction while a normal vector at  $u = v = 1$  is toward a negative  $Z$ -direction. Therefore, there should exist a curve,  $\delta$ , on the surface around which normal vectors change the direction. Any point on  $\delta$  has a degenerate normal vector. In the following discussions, a technique will be presented which applies to cases of interior degenerate normal vectors of a regular parametric surface.

### 3. Detection of interior degenerate normal vectors

The following definitions of cones and maps are quoted from (Kim et al., 1995), and readers are recommended to refer to the article for the details of definitions and algorithms.  $B(u, v)$  has two tangent cones and a normal cone.

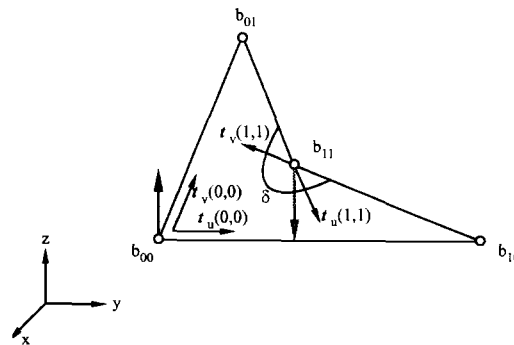


Fig. 2. Curve  $\delta$  where normal vectors degenerate.

**Definition 1.** Tangent cones,  $\mathcal{T}_u$  and  $\mathcal{T}_v$ , and a normal cone,  $\mathcal{N}$ , for  $B(u, v)$  are defined as

$$\mathcal{T}_u = \left\{ p \mid p = \lambda \frac{\partial}{\partial u} B(u, v), \forall \lambda \geq 0, u, v \in [0, 1] \right\}$$

$$\mathcal{T}_v = \left\{ p \mid p = \lambda \frac{\partial}{\partial v} B(u, v), \forall \lambda \geq 0, u, v \in [0, 1] \right\}$$

$$\mathcal{N} = \{ p \mid p = \lambda n, \forall \lambda \geq 0, u, v \in [0, 1] \}$$

where  $p \in \mathbb{E}^3$ , and  $n$  is a normal vector on  $B(u, v)$ .

**Definition 2.** Let  $\sigma$  be a sphere with unit radius centered at the origin of  $\mathbb{E}^3$ . Then, tangent maps,  $T_u$  and  $T_v$ , for  $B(u, v)$  are defined as  $T_u = \mathcal{T}_u \cap \sigma$  and  $T_v = \mathcal{T}_v \cap \sigma$ . A normal map,  $N$ , for  $B(u, v)$  is defined as  $N = \mathcal{N} \cap \sigma$ .

Let  $P^O$  be a plane passing through the origin. A closed region  $S \subset \sigma$  is called *hemispherical* if there exists  $P^O$  placing  $S$  in a half-space.  $S$  is called *spherical convex* if there exists a plane  $P \neq P^O$  such that the projection of  $S$  onto  $P$  is a convex region.

Let  $\delta$  be a curve on  $B(u, v)$ , and  $p \in \delta$  be a point of  $B(u, v)$ . If a normal vector of the surface at  $p$  degenerates, there are normal vectors of  $B(u, v)$  at points in the neighborhood of  $p$  which are parallel but reversed. Therefore, there exists a line passing through the origin and intersecting with the normal map of  $B(u, v)$  twice, when  $B(u, v)$  has an interior degenerate normal vector. On the contrary, if a normal map of  $B(u, v)$  is hemispherical, there does not exist a degenerate normal vector interior to  $B(u, v)$ .

**Definition 3.**  $\Phi$  is a mapping  $\Phi : (S_1, S_2) \Rightarrow S_3$  such that

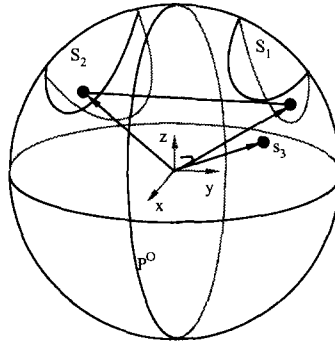
$$S_3 = \left\{ s_3 \mid s_3 = \frac{s_1 \times s_2}{|s_1 \times s_2|}, s_1 \in S_1, s_2 \in S_2, S_1 \subset \sigma, S_2 \subset \sigma \right\}.$$

Note that  $S_3 \subset \sigma$ , since the cross product of two unit vectors is normalized.

**Theorem 1.**  $S_3$  is hemispherical if

1.  $S_1$  and  $S_2$  are spherical convex,
2.  $S_1 \cap S_2 = \emptyset$ , and
3.  $S_1 \cup S_2$  is hemispherical.

**Proof.** Since  $S_1 \cup S_2$  is hemispherical,  $S_1$  and  $S_2$  can be placed on the northern hemisphere as shown in Fig. 3. Since  $S_1$  and  $S_2$  are spherical convex and  $S_1 \cap S_2$  is null, there exists a plane  $P^O$  separating  $S_1$  and  $S_2$  as shown in the figure. Let  $x$  denote a unit vector toward positive  $X$  axis. Then  $s_3 = (s_1 \times s_2) / |s_1 \times s_2|$  is a unit normal vector of a plane defined by  $s_1, s_2$  and the origin. Since  $x \cdot s_3 > 0$ , the angle

Fig. 3. Hemisphericality of  $\Phi$ .

between two vectors is less than  $\pi/2$ . Therefore,  $s_3$  lies on the front hemisphere.  
□

**Lemma 2.** Let  $S_1$  and  $S_2$  be  $T_u$  and  $T_v$  of  $B(u, v)$ , respectively. Then,  $\Phi(T_u, T_v) \supseteq N$ .

**Proof.**  $\Phi(T_u, T_v)$  is defined as a set of cross-products of all combinations of tangent vectors from  $T_u$  and  $T_v$ , whereas  $N$  is defined as a set of all normal vectors which are cross-products of particular combinations of tangent vectors from  $T_u$  and  $T_v$ .  
□

The following corollary is an immediate consequence of the above theorem and lemma, and is stated without a proof.

**Corollary 2.1.**  $N$  is hemispherical, and there does not exist a degenerate normal vector on  $B(u, v)$  if

1.  $T_u$  and  $T_v$  are spherical convex,
2.  $T_u \cap T_v = \emptyset$ , and
3.  $T_u \cup T_v$  is hemispherical.

Since a tangent cone is computed via a convex hull of the control points of hodograph (Kim, 1990; Kim et al., 1995), the condition 1 is automatically satisfied. Therefore, it is possible to decide if a degenerate normal vector does not exist on  $B(u, v)$  by testing the union and intersection between two tangent cones which are represented as sets of planar faces. Fig. 4 shows the surface in Fig. 2 again and its tangent maps. Since the surface is planar, tangent maps are reduced to arcs on a unit circle in  $XY$ -plane. In this example, the union of two tangent maps is not hemispherical, which violates the condition 3 of the corollary.

However, caution should be exercised when applying the corollary since it states only a sufficient condition for the existence of degenerate normal vectors interior to surface. In other words, a surface may not contain a degenerate normal vector even if a condition of the corollary is violated. There are two reasons: First,  $N$  is a

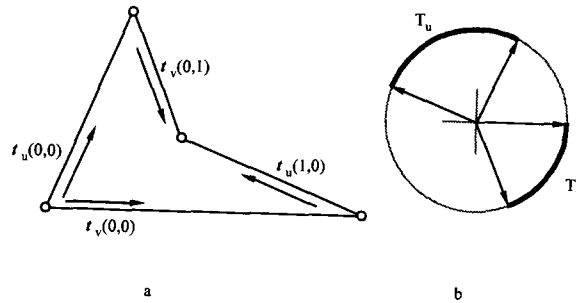


Fig. 4. Non-hemispherical  $T_u \cup T_v$  caused by degenerate normal vectors. (a) Surface with interior degenerate normal vectors. (b) Tangent maps of the surface.

subset of  $\Phi(T_u, T_v)$  as was stated in Lemma 2. Second, the computed tangent cones,  $\hat{T}_u$  and  $\hat{T}_v$ , which satisfy the condition 1 of the corollary are always over-estimation of the true tangent cones,  $T_u$  and  $T_v$ . Readers are recommended to refer to (Kim, 1990; Kim et al., 1995) for the details of the estimation measure of a tangent cone. The corollary only guarantees that there is no degenerate normal vector on a surface if the conditions are met. When Corollary 2.1 is applied to a surface to find if one of the conditions is truly violated, it is recommended that the surface be subdivided a few times and the corollary be applied to each subsurface before making a decision of the existence of a degenerate normal vector on the surface.

#### 4. Conclusions

Normal vectors of a surface are important in various geometric operations. Since a normal vector at a point on a parametric surface is computed by a cross-product of two partial derivatives at the point, even a regular parametric surface may have a degenerate normal vector when the partials at the point are linearly dependent. In this paper, a technique is provided to make sure that there is no degenerate normal vector on a regular parametric surface. However, the conditions to detect a degenerate normal vector is not a necessary-and-sufficient condition. It is necessary to do a number of subdivisions and apply the technique to each subsurface to determine whether the surface does or does not have a degenerate normal vector interior to the surface.

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