Optimization of Hair Beauty in African-American Women

by

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ABSTRACT

The goal of this project is to maximize the beauty of African-American women's hair, which has been determined to be a function of hair straightening, hair styling, and hair care techniques. Specifically, the conflicting desires that these women have in wishing to achieve maximum hair straightening, maximum hair styling capability, and maximum hair health have motivated a rigorous optimization study on this issue. Also, the lack of adequately documented information regarding rigorous, formal optimization techniques that maximize the beauty of black women's hair has also motivated this study. This system-wide optimization problem will first be decomposed into the three aforementioned core areas and examined independently to determine their own optimized solutions. Finally, upon considering all of these areas concurrently on the system-wide level, it is anticipated that an optimized solution will be found that maximizes hair beauty slightly better than the existing, more conservative methods.

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1. INTRODUCTION

Although African-American women are blessed with many attractive features, one of the crowning beauty assets for many black women is their hair. In 2006 alone, approximately $1.9 billion was spent on hair care and beauty products (US Market for Ethnic Hair report). It is clear, therefore, that African-American women are investing in products that augment the beauty of their hair. While each black women has her own unique style, research has revealed that there are three key trends among the vast majority of these women. First, many African-American women desire to have hair that is straightened from its natural, curly state such that their hair appears to be longer. This is important to them because straighter, longer hair often enhances the capability to style and manage their hair. Another key trend that is implied by the first is that black women (as do most women) want to have the freedom to style their hair anyway they choose. Whether it's layered, flipped, braided, twisted, crimped, or dyed, African-American women want to be able to style their hair as much as any other woman. The final important trend among these women is that they wish to have hair that can undergo these first two processes and remain perfectly healthy. However, this last trend is in direct conflict with the other two because many of the straightening and styling techniques used on black women's hair are inherently damaging to the hair. Hence, these tradeoffs among hair straightening, styling, and health provide the motivation for a system-wide study to maximize the beauty of African-American women's hair. This system optimization study is the first of its kind as there is virtually no documented information regarding formal, rigorous techniques used to select the best straightening, styling, and hair care techniques that can provide such an intangible output as hair beauty.

2. EFFECTIVE HAIR LENGTH (HAIR STRAIGHTENING)

By Michael Alexander

2.1 Problem Statement

Although there are some African-American women that are content with their hair in its natural, curly state, the majority of black women desire to have hair that is straight for styling and manageability reasons. These women feel that with better styling and manageability, they will also be able to enhance their beauty. The process for straightening African-American women's hair involves maximizing the effective hair length such that it approximates the actual hair length. Note that the effective length refers to the length of hair as measured from end to end in its coiled state, whereas the actual length refers to the length of hair measured from end to end in its uncoiled state. Typically, the straightening process is carried out through two methods: chemical relaxing and thermal relaxing. In chemical relaxing, a highly caustic chemical solution (usually containing sodium hydroxide or calcium hydroxide) is applied to the hair for a specified amount of time. While in the hair, the relaxer destroys disulfide bonds, which are partly responsible for providing the curl. As these bonds are broken, the hair becomes straighter. Similarly, in thermal relaxing, heat is applied to the hair though a flat iron for a given amount of time. During this process, the heat breaks hydrogen bonds, which are also responsible for providing curl in the hair. Finally, as these bonds are broken, the hair becomes straighter.

Ideally, an African-American woman would desire to maximize both chemical and thermal relaxing in terms of quantity, time, and frequency of application in order to achieve the longest, straightest hair possible. However, there are limits to these processes as excessive chemical and thermal relaxing can lead to hair damage via breakage. It is this tradeoff between the two relaxing techniques and breakage that provides the motivation for this optimization study. Note that this optimization study is the first of its kind as there is virtually no documented information regarding formal, rigorous techniques used to select the best chemical and thermal relaxing methods for a specific African-American female hair type.

2.2 Nomenclature

The following symbols are used for the variables, parameters, and constants involved in this optimization problem:

Symbols	Meaning	Units
m_r	Quantity of chemical relaxer (mass)	kg
t_r	Duration time of chemical relaxer application	s
f_r	Frequency of chemical relaxer application	#applications/week
t_{th}	Duration time of thermal relaxing	s
f_{th}	Frequency of thermal relaxing	#applications/week
T_{th}	Temperature applied during thermal relaxing	K
$\Delta L_{eff,max}$	Maximum change in effective hair length	m
C_1	Exponential argument scaling coefficient	--
C_2	Chemical relaxing scaling coefficient	--
T_{room}	Ambient temperature during thermal relaxing	K
T_{bp}	Boiling point of water	K
k	Thermal conductivity of hair	W/m*K
c_v	Specific heat of water	J/kg*K
L	Actual hair length	m
d_{app}	Apparent circular diameter of hair	μm
A	Area of flat iron plate coverage	m^2
ρ_{hair}	Area density of hair	hairs/cm^2
ρ_{H20}	Mass density of water	kg/m^3
C_3	Chemical relaxer cost sensitivity coefficient	--
C_4	Chemical/thermal relaxing time sensitivity coefficient	--
C_{relax}	Cost of professional chemical relaxing	$

Table 2.2.1 Effective Hair Length Subsystem Nomenclature

2.3 Mathematical Model

2.3.1 Objective Function

The objective function for this subsystem is to maximize the change in the effective hair length of an African-American woman with coarse hair texture through chemical and thermal relaxing. This would cause the overall effective hair length to increase, thus creating the appearance of longer hair. Note that the choice of an African-American woman with the hair texture described above has been made to consider a "worst case" system design scenario.

Prior to any formulations, the following major assumptions have been made:

1. The woman maintains a healthy diet and therefore does not negatively affect the hair growth rate.

2. The woman's scalp is healthy, i.e. there are no scrapes, cuts, or abrasions that might pose harm to her when exposed to harsh chemical relaxers.
3. The woman is not taking any drugs (medical or otherwise) that might negatively impact the hair growth rate.
4. Only one type/formulation of chemical relaxer (NaOH) is being used.

In deriving the objective function, exponential functions have been chosen to approximate the phenomena caused by the relaxing treatments. This is a good initial approach to modeling the problem for the following reasons:

1. Most phenomena in nature are described by nonlinear functions, and, in many cases, end up being expressed as a linear combination of exponentials when solved analytically via ordinary differential equation theory.
2. Similar expressions have been used in other effective length and hair straightening efficacy models (Quadlfieg, 2003).
3. The nature of the effects of the relaxing techniques is such that it causes the change in effective hair length to approach an asymptotic value, i.e., a woman will never be able to straighten her hair completely.

Using this framework, an exponential expression similar to that of a step input function has been generated describing the "positive" effects of chemical and thermal relaxing:

$$\frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th}}\right) \qquad \text{Equation 2.3.1}$$

Observe that in this expression, all variables are inputs into the argument of the exponential function for simplicity. Also, the general trends between the input variables and the change in effective hair length are captured: as each of the variables increase (more chemical or thermal relaxing), the change in the effective hair length increases sharply until it reaches a limiting value, which, when added to the initial effective hair length, will result in the actual hair length.

Similarly, the "negative" effects of the relaxing techniques are captured in another exponential expression. Here, it is desired to capture the fact that as the hair is subjected to more relaxing treatments, the fracture strength actually decreases over time. This decrease in fracture strength causes the hair to be more prone to breakage, which, by definition, decreases the effective hair length. By extrapolating from a plot in Akkermans, 2004, and incorporating the effects of chemical relaxing, the following relationship has been developed for fracture strength:

$$\sigma_{frac} = -.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r \qquad \text{Equation 2.3.2}$$

Note that the coefficient for the first term in the above relation has been extrapolated from the slope of a linear plot and the coefficient of the second term is based on the assumption that the effects of chemical relaxing will be on the order of ten times more dominant than that of thermal relaxing. Using this formulation, the "negative" effects of the relaxing techniques can be described by:

$$-L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r}) \qquad \text{Equation 2.3.3}$$

In this equation, note that when no relaxing is performed (i.e. the inputs are zero), there is no loss of hair length. However, as the amount of relaxing increases, more hair is lost, and, if performed without "bound", could lead to complete hair loss. The total change in the effective hair length can then be described by combining Equations 2.3.1 and 2.3.3:

$$\Delta L_{eff} = \frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th}}\right) - L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r}) \qquad \text{Equation 2.3.4}$$

After considering the overall system optimization, however, it has been noted that the exponential expression accounting for damage effects (Equation 2.3.3) is redundant and does not need to be added to this subsystem. This is because another subsystem (see "Hair Damage" section) directly addresses issues relating to damage in its own function. Therefore, the objective function has been reduced to the following simpler expression:

$$\Delta L_{eff} = \frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th}}\right) \qquad \text{Equation 2.3.5}$$

Since no tradeoff exists explicitly in this objective function, additional constraints based on physical phenomena in the subsystem will have to be added to provide the necessary tradeoff.

Through experimenting with this formulation, it has been found that two parameters should be added to scale the function output accordingly to ensure that it has physical meaning. The first parameter that has been added scales all exponential arguments such that the product of these arguments is always greater than or equal to unity (knowing that all variables are strictly positive during optimization):

$$\Delta L_{eff} = \frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-C_1 m_r t_r f_r} - e^{-C_1 T_{th} t_{th} f_{th}}\right) \qquad \text{Equation 2.3.6}$$

This is necessary because if the exponential arguments are significantly less than unity, the exponential itself will evaluate to nearly unity. If this occurs, the above equation would yield a result stating that the change in effective hair length is nearly zero, which is not physically possible for strictly positive variables. It should be noted that an initial value of 10^4 has been assigned to this parameter based on the minimum values that the variables could assume during optimization (see Table 2.3.1). From here, a parameter has been added to scale the chemical relaxing variables such that they would have a more significant impact on the change in the effective hair length than the thermal relaxing variables:

$$\Delta L_{eff} = \frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-C_1 C_2 m_r t_r f_r} - e^{-C_1 T_{th} t_{th} f_{th}}\right) \qquad \text{Equation 2.3.7}$$

Physically, the inclusion of this parameter makes sense because chemical relaxing is inherently more effective in changing hair length than thermal relaxing. Initially, a value of 2 has been assigned to this parameter based on the assumption that chemical relaxing would have twice the impact on hair straightening compared to thermal relaxing.

Finally, since it is desired to maximize the above objective function while solving it in a negative-null form, the equation is negated and appears in the following representation to be minimized:

$$-\Delta L_{eff} = -\frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-C_1 C_2 m_r t_r f_r} - e^{-C_1 T_{th} t_{th} f_{th}}\right) \qquad \text{Equation 2.3.8}$$

2.3.2 Constraints

Like the objective function, several of the constraints used by this model have been modified based on experimentation with the formulations and consideration of the system-wide optimization problem. Additionally, some constraints have been completely eliminated or changed to reduce the order of computational complexity. Since it is possible that some of these constraints may still be legitimate but need further investigation, they are included in the Appendix. Nevertheless, the constraints that follow sufficiently capture the essence of the effective hair length design problem.

The first constraint for this subsystem is based on energy balance during thermal relaxing. Since the only process being considered is the heating up of the hair, the (rate-based) energy equation reduces to

$$\dot{E}_{in} = \Delta \dot{E}_{internal} \qquad \text{Equation 2.3.9}$$

where the term on the left side of the equation describes all energy inputs into the system and the term on the right side of the equation describes the change in internal energy of the system. The only energy input into the system is conduction of the hair due to the flat iron, which, for a single strand of hair, is described by:

$$\dot{E}_{in} = \frac{2\pi L k\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\frac{0.5 d_{app}}{0.5 d_{app} - 0.5x10^{-6}}\right)} \qquad \text{Equation 2.3.10}$$

Observe that this expression is based on conduction of a "hollow tube". The change in internal energy of the system is due to a temperature rise of the water within the hair, which is described by:

$$\Delta \dot{E}_{\text{int}\,ernal} = \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 Lc_v(T_{bp} - T_{room})}{t_{th}} \qquad \text{Equation 2.3.11}$$

Substituting Equations 2.3.10 and 2.3.11 into 2.3.9 provides the necessary energy balance constraint:

$$h_1 = \frac{2\pi Lk\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\dfrac{0.5d_{app}}{0.5d_{app} - 0.5x10^{-6}}\right)} - \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 Lc_v(T_{bp} - T_{room})}{t_{th}} = 0$$

Equation 2.3.12

The first inequality constraint limits the amount and frequency of chemical relaxing performed by cost. In general, it is known that these two input variables are inversely proportional to one another; if more chemical relaxer is applied during a given treatment, then the treatments must be made less frequently, and vice versa. A cost sensitivity parameter, which has been initially assigned a value of 1, is added to the equation such that the product of this constant and the two variables will result in a total cost for professional chemical relaxing. This is expressed by:

$$g_1 = C_3 m_r f_r - C_{relax} \leq 0 \qquad \text{Equation 2.3.13}$$

Another inequality constraint limits the values of the product of the duration time and frequency of a chemical relaxing application. This would create the practical and necessary condition that the time and frequency of chemical relaxing are inversely proportional to each other, i.e., as chemical relaxation time increases, the chemical relaxation frequency decreases, and vice versa. Note that the limiting values for each of these variables (Table 2.3.1) are used to construct the upper and lower bounds for this relationship. Specifically, the maximum value for the chemical relaxation time is multiplied by the minimum value for the chemical relaxation frequency to obtain one bound, and vice versa for the other bound. The values are assigned to the appropriate bounds (upper or lower) based on the relative magnitudes of these products. Finally, a sensitivity coefficient has been added to this equation to ensure that:

1. The duration time for chemical relaxing is weighted heavier than the frequency, assuming that this contributes more to both hair straightening and damage, and
2. In the event of activity, this constraint would not result in the trivial case of hitting upper and lower bounds (which is a possibility based on the constraint's construction). This procedure produces the following expressions:

$$g_2 = -C_4 t_r f_r - 0.0002478 \le 0 \qquad \text{Equation 2.3.14}$$

$$g_3 = C_4 t_r f_r - 0.0002484 \le 0 \qquad \text{Equation 2.3.15}$$

where C_4 has initially been assigned a value of 1.5.

Performing a similar procedure for the thermal relaxation time and frequency variables yields the following constraints:

$$g_4 = -C_4 t_{th} f_{th} + 0.000002481 \le 0 \qquad \text{Equation 2.3.16}$$

$$g_5 = C_4 t_{th} f_{th} - 0.000011574 \le 0 \qquad \text{Equation 2.3.17}$$

Another inequality constraint limits the values of the product of the quantity of chemical relaxer and temperature of thermal relaxing. Like the time and frequency variables, this creates the practical and necessary condition that the quantity of chemical relaxer and temperature of thermal relaxing are inversely proportional to each other, i.e., as more chemical relaxer used, less heat can be applied to the hair, and vice versa. Furthermore, such a constraint provides the key tradeoff between chemical and thermal relaxing that enables an optimization study to be performed for hair straigtening. To formulate the constraints, a method similar to that used for the time and frequency variables has been employed and results in the following equations:

$$g_6 = -C_2 m_r T_{th} + 23.27 \le 0 \qquad \text{Equation 2.3.18}$$

$$g_7 = C_2 m_r T_{th} - 58.5 \le 0 \qquad \text{Equation 2.3.19}$$

The remaining inequality constraints are all practical constraints, which set simple upper and lower bounds. All of the following constraints are based on research found in journals, on the Internet, or through interviews with hairstylists that are referenced at the end of this document:

$$g_8 = 0.05 - m_r \le 0$$
$$g_9 = m_r - 0.15 \le 0$$
$$g_{10} = 600 - t_r \le 0$$
$$g_{11} = t_r - 1200 \le 0$$
$$g_{12} = 1/8 - f_r \le 0$$
$$g_{13} = f_r - 1/4 \le 0$$
$$g_{14} = 390 - T_{th} \le 0$$
$$g_{15} = T_{th} - 475 \le 0$$
$$g_{16} = 1 - t_{th} \le 0$$
$$g_{17} = t_{th} - 3 \le 0$$
$$g_{18} = 1/2 - f_{th} \le 0$$
$$g_{19} = f_{th} - 7 \le 0$$

2.3.3 Design Variables and Parameters

The table below indicates the design variables involved in the objective function and constraints along with typical values for these quantities. All of the values are based on research from journal articles, the Internet, or hairstylist interviews. Assuming that none of the inequality constraints are active and that all equality constraints are active, the model has potentially five degrees of freedom.

Symbols	Meaning	Typical Values
m_r	Quantity of chemical relaxer (mass)	0.05-0.15 kg
t_r	Duration time of chemical relaxer application	600-1200 s
f_r	Frequency of relaxer application	0.125-0.25 applications/week
t_{th}	Duration time of thermal relaxing	1-3 s
f_{th}	Frequency of thermal relaxing	0.25-7 applications/week
T_{th}	Temperature applied during thermal relaxing	390-475 K

Table 2.3.1 Effective Hair Length Subsystem Variables

The next table indicates the design parameters involved in the objective function and constraints along with typical values for these quantities. Again, all of the values (unless stated otherwise earlier in this document) are based on research from journal articles, the Internet, or hairstylist interviews.

Symbols	Meaning	Typical Values
$\Delta L_{eff,max}$	Maximum change in effective hair length	0.050 m
C_1	Exponential argument scaling coefficient	10000
C_2	Chemical relaxing scaling coefficient	2
T_{room}	Ambient temperature during thermal relaxing	300 K
T_{bp}	Boiling point of water	373 K
k	Thermal conductivity of hair	0.040 W/m*K
c_v	Specific heat of water	4184 J/kg*K
L	Actual hair length	0.150 m
d_{app}	Apparent circular diameter of hair	73.6 μm
A	Area of flat iron plate coverage	0.00194 m^2
ρ_{hair}	Area density of hair	187 hairs/cm^2
ρ_{H20}	Mass density of water	1000 kg/m^3
C_3	Chemical relaxer cost sensitivity coefficient	1
C_4	Chemical/thermal relaxing time sensitivity coefficient	1.5
C_{relax}	Cost of professional chemical relaxing	$50

Table 2.3.2 Effective Hair Length Subsystem Parameters

2.3.4 Summary Model

After carefully examining the subsystem model, it has been determined that constraints g_2, g_4 and g_6 could be eliminated as their corresponding upper bounds g_3, g_5, and g_7 would sufficiently provide the necessary relationships between these variables. Therefore, the subsystem design optimization problem, with the remaining constraints renumbered can be summarized accordingly in standard, negative null form:

$$\min\ -\Delta L_{eff} = -\frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-C_1 C_2 m_r t_r f_r} - e^{-C_1 T_{th} t_{th} f_{th}}\right)$$

subject to $g_1 = C_4 m_r f_r - C_{relax} \le 0$

$$g_2 = C_4 t_r f_r - 0.0002484 \le 0$$

$$g_3 = C_4 t_{th} f_{th} - 0.000011574 \le 0$$

$$g_4 = C_2 m_r T_{th} - 58.5 \le 0$$

$$g_5 = 0.05 - m_r \le 0$$

$$g_6 = m_r - 0.15 \le 0$$

$$g_7 = 600 - t_r \le 0$$

$$g_8 = t_r - 1200 \le 0$$

$$g_9 = 1/8 - f_r \le 0$$

$$g_{10} = f_r - 1/4 \le 0$$

$$g_{11} = 390 - T_{th} \le 0$$

$$g_{12} = T_{th} - 475 \le 0$$

$$g_{13} = 1 - t_{th} \le 0$$

$$g_{14} = t_{th} - 3 \le 0$$

$$g_{15} = 1/2 - f_{th} \le 0$$

$$g_{16} = f_{th} - 7 \le 0$$

$$h_1 = \frac{2\pi L k\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\dfrac{0.5 d_{app}}{0.5 d_{app} - 0.5x10^{-6}}\right)} - \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 L c_v (T_{bp} - T_{room})}{t_{th}} = 0$$

2.4 Model Analysis

2.4.1 Monotonicity Analysis

In performing the monotonicity analysis, the h_1 constraint is first eliminated from the optimization problem (assuming it is active). Solving for t_{th} explicitly and then substituting into related equations gives the following optimization problem:

$$\min\ -\Delta L_{eff} = -\frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-C_1 C_2 m_r t_r f_r} - e^{-C_1 \frac{K_1}{K_2 (T_{th} - T_{toom})} T_{th} f_{th}}\right)$$

subject to $g_1 = C_4 m_r f_r - C_{relax} \le 0$

$$g_2 = C_4 t_r f_r - 0.0002484 \le 0$$

$$g_3 = C_4 \frac{K_1}{K_2 (T_{th} - T_{room})} f_{th} - 0.000011574 \le 0$$

$$g_4 = C_2 m_r T_{th} - 58.5 \le 0$$

$$g_5 = 0.05 - m_r \le 0$$
$$g_6 = m_r - 0.15 \le 0$$
$$g_7 = 600 - t_r \le 0$$
$$g_8 = t_r - 1200 \le 0$$
$$g_9 = 1/8 - f_r \le 0$$
$$g_{10} = f_r - 1/4 \le 0$$
$$g_{11} = 390 - T_{th} \le 0$$
$$g_{12} = T_{th} - 475 \le 0$$
$$g_{13} = 1 - \frac{K_1}{K_2 (T_{th} - T_{room})} \le 0$$
$$g_{14} = \frac{K_1}{K_2 (T_{th} - T_{room})} - 3 \le 0$$
$$g_{15} = 1/2 - f_{th} \le 0$$
$$g_{16} = f_{th} - 7 \le 0$$

where K_1 and K_2 are used for convenience to account for the constants in the energy equation and are equal to 1.89×10^{-1} and 7.59×10^{-4} respectively. From here, a monotonicity table is able to be constructed as illustrated below:

	m_r	t_r	f_r	T_{th}	f_{th}
f	-	-	-	+	-
g_1	+		+		
g_2		+	+		
g_3				-	+
g_4	+			+	
g_5	-				
g_6	+				
g_7		-			
g_8		+			
g_9			-		
g_{10}			+		
g_{11}				-	
g_{12}				+	
g_{13}				+	
g_{14}				-	
g_{15}					-
g_{16}					+

Table 2.4.1 Monotonicity Table for Change in Effective Hair Length

Applying monotonicity principles, activities can then be identified and eliminated. By Monotonicity Principle 1 (MP1) with respect to m_r, it is known that $\{g_1, g_4, g_6\}$ is conditionally active. To determine which of the constraints are active, plots of m_r within the feasible domain have been generated in MATLAB and compared to one another. Based on these results, g_4 has been identified as the dominant constraint and hence it is active. Next, by MP1 with respect to t_r, it is known that $\{g_2, g_8\}$ is conditionally active. Using a dominance argument similar to that for m_r, g_2 has been identified as active. At this point, the active constraints are eliminated from the optimization problem and the monotonicity table reduces to the following:

	f_r	T_{th}	f_{th}
f		U	U
g_1	+	-	
g_3		-	+
g_5		+	
g_6		-	
g_7	+		
g_8	-		
g_9	-		
g_{10}	+		
g_{11}		-	
g_{12}		+	
g_{13}		+	
g_{14}		-	
g_{15}			-
g_{16}			+

Table 2.4.2 Reduced Monotonicity Table for Change in Effective Hair Length

Observe that f_r no longer appears in the objective function. Now, by Monotonicity Principle 2 (MP2) with respect to f_r, it has been found that this variable is irrelevant to the solution process and can be eliminated (none of the remaining constraints can bound this variable). The resulting optimization problem now has two degrees of freedom (T_{th} and f_{th}) with monotonicities that appear to vary in the objective function (i.e., the function is not always increasing or decreasing). Since it is relatively difficult to explicitly identify optimality conditions for this exponential expression, this problem is solved using MATLAB's built in Sequential Quadratic Programming (SQP) solver, "fmincon".

2.5 Optimization Study

In numerically determining the optimal point for this problem, a code has been written in MATLAB that makes use of its "fmincon" function (Appendix B.1). Initially, no scaling has been used despite the differences in the orders of magnitudes of the variables. This has been done intentionally to determine the efficacy and robustness of the subsystem model in a direct way. Note that the constraint tolerances have been set to 1×10^{-10} to generate higher accuracy in the algorithm. The key items that are reported in the output include the optimal point (x*), the function value at the optimal point (f*), constraint activity, and the Lagrange multiplier values. This information is highlighted in Tables 2.5.1-2.5.4 for various starting points in the algorithm. Note that for all starting points, an "exit flag 5" has been reported from the algorithm. This indicates that the solver terminated because it could not find a numerically significant slope direction, i.e., the derivatives of the function were very small. Additionally, it should be observed that the function values are negative only because the objective function has been expressed in negative-null form.

	x*	f*	Active Constraints	Lagrange Multipliers
m_r	7.09E-02	-3.02E-02	h_1	-5.76E-06
t_r	6.00E+02		g_2	1.86E+01
f_r	2.76E-07		g_4	7.94E-05
T_{th}	4.13E+02			
t_{th}	2.22E+00			
f_{th}	3.30E-06			

Table 2.5.1 MATLAB "fmincon" Results for Starting Point $x_0=(0,0,0,0,0,0)^T$

	x*	f*	Active Constraints	Lagrange Multipliers
m_r	7.09E-02	-3.02E-02	h_1	-5.77E-06
t_r	6.00E+02		g_2	1.86E+01
f_r	2.76E-07		g_4	7.94E-05
T_{th}	4.12E+02			
t_{th}	2.22E+00			
f_{th}	3.38E-06			

Table 2.5.2 MATLAB "fmincon" Results for Starting Point $x_0=(1,1,1,1,1,1)^T$

	x*	f*	Active Constraints	Lagrange Multipliers
m_r	7.50E-02	-3.05E-02	h_1	-1.55E-05
t_r	6.00E+02		g_2	1.94E+01
f_r	2.76E-07		g_4	7.94E-05
T_{th}	3.90E+02			
t_{th}	2.77E+00			
f_{th}	2.77E-06			

Table 2.5.3 MATLAB "fmincon" Results for Starting Point $x_0=(10,10,10,10,10,10)^T$

	x*	f*	Active Constraints	Lagrange Multipliers
m_r	7.50E-02	-3.05E-02	h_1	-1.55E-05
t_r	6.00E+02		g_2	1.94E+01
f_r	2.76E-07		g_3	5.09E-09
T_{th}	3.90E+02		g_4	8.28E-05
t_{th}	2.77E+00			
f_{th}	2.78E-06			

Table 2.5.4 MATLAB "fmincon" Results for Starting Point x_0=Lower Bounds

Based on the above results, it can be seen that some of the trends of the monotonicity analysis are in agreement with the numerical results. Specifically, it is evident that both g_2 and g_4 are identified as active constraints, along with the obvious activity of h_1. Since these are the only active constraints in Tables 2.5.1-2.5.3, this might indicate that the model has three degrees of freedom, which would not coincide with the two degree of freedom system identified through monotonicity analysis. However, in Table 2.5.4, g_3 also becomes active, which would then reduce the model to a two degree of freedom system as determined through monotonicity analysis. Because each starting point has produced different results for both the minimum and the minimizers, a scaling procedure has been attempted (Appendix B.1) to determine the true optimum. This scaling procedure, however, fails to produce feasible results, which may be due to numerical function sensitivities that have been built into the model through "tweaking" parameters. The scaling issues may also be due the significant differences in the Lagrange multipliers (and hence sensitivities) for each constraint. Note that the Lagrange multiplier for g_2 is 10^6 times larger than multipliers for h_1 and g_4 and nearly 10^{10} times large than the multiplier for g_3. Therefore, the best assumption that can be made is that the result for the lower bound starting point (Table 2.5.4) is "optimal". This is reasonable because:

1. The solution is in agreement with monotonicity analysis, which predicts a two degree of freedom system with confirmed activities for g_2 and g_4.
2. The solution provides the lowest minimum despite having an "exit flag 5".
3. The starting point for this solution is the most realistic out of all selected points.

Although global convergence is not definitive, this model is still quite meaningful. First, the optimal value indicates a change in effective length of nearly 3.05 cm, which is realistic based on the maximum value possible (5.0 cm) for this model. The suggested chemical relaxing process involves about 0.075 kg of relaxer applied for just above 10 minutes (600 s) every 6 weeks. This is slightly more aggressive in terms of quantity (0.05 kg) and frequency (1 application/8 weeks) than the traditional chemical straightening process. The suggested thermal relaxing process involves just above 390 K (117°C) of heat applied for just over 2.50 s (per strand(s) of hair, with a flat iron) between 1-2 times per week. This is a somewhat conservative, yet expected result since more chemical relaxing requires less thermal relaxing. Therefore, the design model that has been developed does in fact capture the essence of the hair straightening process with interesting tradeoffs.

2.6 Parametric Study

For this part of the subsystem analysis, an investigation has been performed to examine the effect of perturbations in the parameter values on the optimal solution. Since the active constraints include the chemical "scaling" coefficient (C_2) and the time/frequency sensitivity coefficient (C_4), these parameters have been selected for the study. The value for C_2 has been varied from 1 to 3 in increments of 0.5, where C_2=2 is the nominal value. Note that no further values can be examined beyond this point since the model becomes infeasible. Based on the results, this parameter has little effect on the optimal value and the majority of optimizer values for a given solution. However, Figures 2.6.1-2.6.2 indicate that some significant behavior has been found in the optimizer values for m_r and f_{th}.

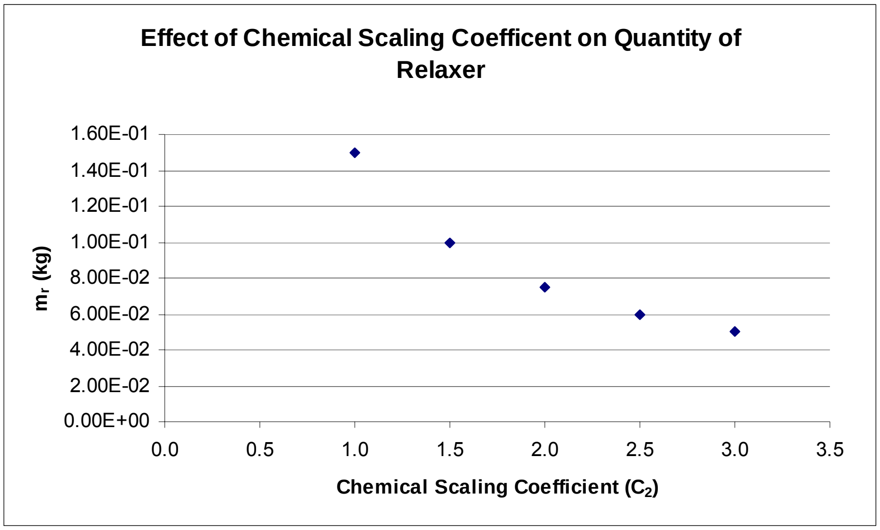

Figure 2.6.1 Effect of Chemical Scaling Coefficient on Quantity of Relaxer

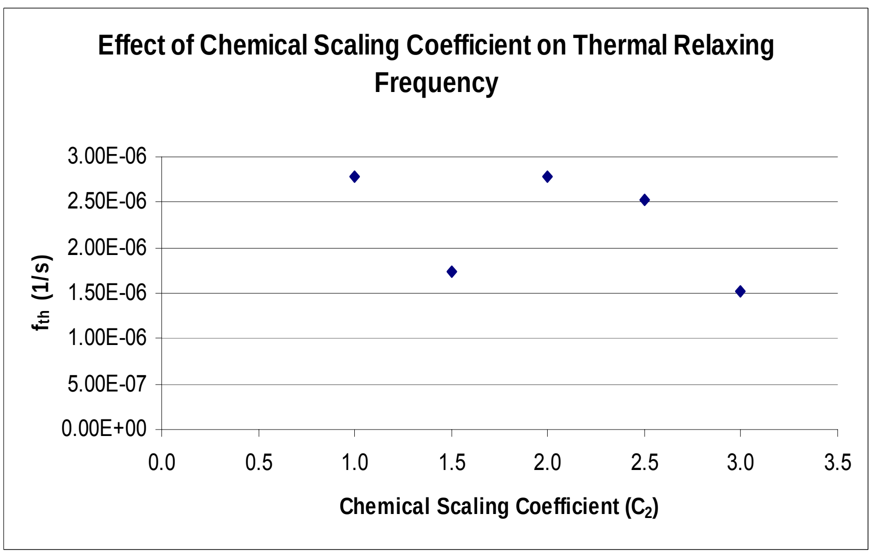

Figure 2.6.2 Effect of Chemical Scaling Coefficient on Thermal Relaxing Frequency

Specifically, it is evident from Figure 2.6.1 that has the chemical scaling coefficient increases, the optimizer value for the quantity of relaxer decreases from its upper bound (0.15 kg) to its lower bound (0.05 kg) in a hyperbolic fashion. This trend implies that making this "scaling" coefficient too high or too low will cause this variable to move to its lower and upper bounds, respectively. Therefore, it is best to use a value between 1 and 3 such that the model will not produce trivial results.

In Figure 2.6.2, it is evident that the optimizer value for the frequency in thermal relaxing varies as the chemical scaling coefficient increases, but in no apparent pattern. Therefore, no definitive conclusion can be made about the nature of the adjustment of the chemical scaling coefficient for this optimizer.

The parametric study for C_4 indicates that the only feasible values for this parameter are 1, 1.5, and 2, where the nominal value is C_4=1.5. Because this range of values is small, a meaningful plot of this parameter's effects on the optimum and optimizers is not practical and hence not included. Nevertheless, this investigation has revealed that the values for both the optimum and the optimizers change radically for each value of the parameter. The only conclusion that can be made regarding this parameter is that better modeling of subsystem constraints associated with it could improve its range of feasible values.

3. HAIR STYLING

By Tahira Reid

3.1 Problem Statement

Among the many hair styles that African-American women enjoy wearing, the following three will be included in the scope of our project: permanent dye, thermal hairstyling (ex. Flat irons and curling irons), and braids with extensions.

The permanent dye process consists of applying a specified volume of dye solution to hair. During the first visit, dye would be applied to the entire hair. Then retouches would be applied to the hair that has grown over a period of time called *new growth*. Thermal styling involves the use of the flat-irons and curling irons. The flat ironing process consists of the application of a normal ("clamping") force on the hair while gliding it in the direction from the scalp to the ends of the hair. Flat irons can also be manipulated to created large voluminous curls. Curling irons involve a similar application of a clamping force and the hair is held and rolled into the barrel shaped curling iron. The diameter of the barrel determines the size of the curl desired. Lastly braiding is a styling technique that has its origins in Africa. For this problem, we will be focusing on "box-braids" so named because the hair is sectioned into a large number of box-shaped sections containing a set amount of hairs. These boxed out sections will then receive the addition of synthetic hair which is anchored to the hair using a special braiding technique and then is mixed in with the client's own hair. The addition of extensions often adds length and versatility to the style. It also gives the braids a more uniform appearance.

Within this subsystem, the anticipated tradeoffs exist between the 3 styling techniques. Braiding is often used as a means for growing out the hair and giving the client a break from having to style hair daily. However, braids may not be recommended for individuals with low hair density and fine hair. It can be damaging to the hair if braided too tight or too small. Dying relaxed hair in general is damaging to the hair and thermal heat styles exacerbate problems if used repeatedly. These three styling techniques can be problematic on unhealthy hair.

3.2 Nomenclature

Below is a complete list of symbols for design variables and parameters that will be used to analyze the styling subsystem.

Symbols	Meaning	Units
t_{dye}	Duration time of dye application	min
f_{dye}	Time between dye application	applications/week
V_{dye}	Volume of dye	mL
l_{braid}	Length of braids	cm
D_{braid}	Diameter of braid	cm
f_{braid}	How often client gets hair braided	applications/week
f_{th}	Frequency at which client has thermal styles done at a salon	styling/week
T_{th}	Temperature setting during thermal styling	K
l_{hair}	Length of hair client starts process with	cm

Table 3.2.1 Hair Styling Subsystem Nomenclature

3.3 Mathematical Model

3.3.1 Objective Function

The objective function for this system is to:

$$\min\ f(\mathbf{x}) = -[(1-e^{-t_{dye} f_{dye} V_{dye}}) + (1-e^{-l_{braid} D_{braid} f_{braid}}) + (1-e^{-(f_{th} T_{th})})]$$

where

$$\mathbf{x} = (t_{dye}, V_{dye}, f_{dye}, l_{braid}, D_{braid}, f_{braid}, f_{th}, T_{th})^T$$

$$\mathbf{p} = l_{hair}$$

The main development driver behind this model was to aim for simplicity and ensure that expected behavior between the individual variables was captured in the constraints.

3.3.2 Constraints

One of the main changes that have occurred since the last report was a complete revamp of the constraints being used. A key driver in the development of these constraints was to capture the physical tradeoffs that actually exist in the physical system. For braiding, there are some tradeoffs that exist between the size of the braids, length of braids, and the frequency at which one would have their hair re-braided. The general relationship that exists between these variables is as follows: as $\mathbf{l_{braid}}$ ↑, then $\mathbf{D_{braid}}$ should ↓ and as $\mathbf{D_{braid}}$ should ↓ then $\mathbf{f_{braid}}$ should ↓.

The constraints g_1 and g_2 define the relationship between the diameter of the braid and the length of the braid. An upper and lower limit was placed on the size of the braids which are captured by multiplying those limits. The constraint is essentially ensuring that the values obtained from the optimization model make physical sense and will in fact add to beauty.

$$g_1 : D_{braid} l_{braid} - 24.06 C_{2s} \leq 0 \qquad \text{Equation 3.3.15}$$

Likewise, the constraints g_3 and g_4 were used to relate the diameter of a bread to the frequency at which one gets a braid. In a practical sense, the smaller the braid size, the longer the style will actually last. In addition, smaller braid size can easily imply that a significant amount of time will be invested for doing the style. Some braid styles can take a braider 12 hours to do, depending on the size. Smaller braids also mean more cost to the client (which is not being considered here).

Hence, when most people get braids of this kind, they want to keep it in for as long as possible. Based on specified values, the constraints below were defined as shown.

Next, constraints were developed to relate the frequency of thermal styling at home (f_{th}) with the typical temp range of hot hair appliances (T_{th}). The constraints g_2 define the frequency at which one would style their hair at home as well as the possible temperature range of the curling/flat iron. It is reasonable to assume that clients doing their hair at home may not get the same quality of results using their home appliance as they would at a salon. This is true because the temperature range of home appliances can't always attain the ones that professional curling irons/flat irons can achieve. In addition, purchasing the high quality irons are an option, but one that is quite expensive. Therefore, the constraints below relate frequency of thermal styling at home to a fraction of the temperature one can achieve at the hair dresser.

$$g_2 : f_{th}T_{th} - 196.5C_{6s} \leq 0 \qquad \text{Equation 3.3.16}$$

The remaining constraints consist of simple upper and lower bounds based on practice in industry and from references.

$$g_3 : -t_{dye} + 15 \le 0$$
$$g_4 : t_{dye} - 30 \le 0$$
$$g_5 : -f_{dye} + 1/52 \le 0$$
$$g_6 : f_{dye} + 1/4 \le 0$$
$$g_7 : -V_{dye} + 29.6 \le 0$$
$$g_8 : V_{dye} - 118.3 \le 0$$
$$g_9 : -l_{braid} + 15.24 \le 0$$
$$g_{10} : l_{braid} - 60.96 \le 0$$
$$g_{11} : -D_{braid} + 0.3175 \le 0$$
$$g_{12} : D_{braid} - 1.5785 \le 0$$
$$g_{13} : -f_{braid} + 1/52 \le 0$$
$$g_{14} : f_{braid} - 1/4 \le 0$$
$$g_{15} : -f_{th} + 1/8 \le 0$$
$$g_{16} : f_{th} - 1 \le 0$$
$$g_{17} : -T_{ts} + 386 \le 0$$
$$g_{18} : T_{ts} - 483 \le 0$$

3.3.3 Design Variables and Parameters

In order to make the best recommendations to the client, it will be necessary to assess the feasibility of implementing certain styles and processes on the client's hair, while keeping hair health in mind. The key design variables and parameters will include the following:

Symbols	Meaning	Typical Values
t_{dye}	Duration time of dye application	min
f_{dye}	Time between dye application	applications/week
V_{dye}	Volume of dye	mL
l_{braid}	Length of braids	cm
D_{braid}	Diameter of braid	cm^2
f_{braid}	How often client gets hair braided	applications/week
f_{th}	Frequency at which client has thermal styles done at a salon	styling/week
T_{th}	Temperature setting during thermal styling	K

Table 3.3.3 Hair Styling Subsystem Variables

Symbols	Meaning	Typical Values
l_{hair}	Length of hair client starts process with (hypothetical)	7.62 cm (3 inches)

Table 3.3.4 Hair Styling Subsystem Parameters

3.3.4 Summary Model

In closing, the model can be summarized as follows:

$$\min\ f(\mathbf{x}) = -[(1-e^{-t_{dye} f_{dye} V_{dye}}) + (1-e^{-l_{braid} D_{braid} f_{braid}}) + (1-e^{-(f_{th} T_{th})})]$$

where

$$\mathbf{x} = (t_{dye}, V_{dye}, f_{dye}, l_{braid}, D_{braid}, f_{braid}, f_{th}, T_{th})^T$$

$$\mathbf{p} = l_{hair}$$

subject to:

$$g_1 : D_{braid} l_{braid} - 24.06 C_{2s} \le 0$$

$$g_2 : f_{th} T_{th} - 196.5 C_{6s} \le 0$$

$$g_3 : -t_{dye} + 15 \le 0$$

$$g_4 : t_{dye} - 30 \le 0$$

$$g_5 : -f_{dye} + 1/52 \le 0$$

$$g_6 : f_{dye} + 1/4 \le 0$$

$$g_7 : -V_{dye} + 29.6 \le 0$$

$$g_8 : V_{dye} - 118.3 \le 0$$

$$g_9 : -l_{braid} + 15.24 \le 0$$

$$g_{10} : l_{braid} - 60.96 \le 0$$

$$g_{11} : -D_{braid} + 0.3175 \le 0$$

$$g_{12} : D_{braid} - 1.5785 \le 0$$

$$g_{13} : -f_{braid} + 1/52 \le 0$$

$$g_{14} : f_{braid} - 1/4 \le 0$$

$$g_{15} : -f_{th} + 1/8 \le 0$$

$$g_{16} : f_{th} - 1 \le 0$$

$$g_{17} : -T_{ts} + 386 \le 0$$

$$g_{18} : T_{ts} - 483 \le 0$$

3.4 Model Analysis

3.4.1 Monotonicity Analysis (MA)

Implementing the MA consisted of first identifying how each of the variables affected the behavior of the objective function. It could be seen by differentiating the objective that the function is primarily decreasing relative to most of the variables, except for f_{braid} and T_{th} where these variables influence it to increase.

	t_{dye}	f_{dye}	V_{dye}	l_{braid}	D_{braid}	f_{braid}	f_{th}	T_{th}
f	-	-	-	-	-	-	-	-
g_1				+	+			
g_2							+	+
g_3	-							
g_4	+							
g_5		-						
g_6		+						
g_7			-					
g_8			+					
g_9				-				
g_{10}				+				
g_{11}					-			
g_{12}					+			
g_{13}						-		
g_{14}						+		
g_{15}							-	
g_{16}							+	
g_{17}								-
g_{18}								+

Table 3.4.1 Monotonicity Table for Styling Subsystem

The monotonicity analysis reveals the following:

- The model is well-bounded. Each of the variables that are present in the objective function are either bounded below or above by an inequality constraint or simple upper and lower bounds.
- By MP1, the follow constraints are active:

- g_1 or g_{10} is active wrt l_{braid}
- g_1 or g_{12} is active wrt D_{braid}
- either g_{14} active wrt f_{braid}
- g_2 or g_{16} is active wrt. f_{th}
- g_2 or g_{18} is active wrt. T_{th}
- g_4 is active wrt t_{dye}
- g_6 is active wrt f_{dye}
- g_8 is active wrt V_{dye}

From this analysis, the values for the dye design variables (t_{dye}, f_{dye}, V_{dye}) and f_{braid} can be obtained directly from constraint activity. Thus, these variables can be eliminated. The remaining 4 variables can then be assessed using fmincon.

3.5 Optimization Study

The analysis in MatLab provided some results consistent with those found in the monotonicity analysis. Table 3.4.2 shows the results of the analysis with a starting point of the x_0 vector being equal to 1. Here, we see more activity in the constraints and the addition of one more Lagrange Multiplier.

x*		f*	Active Constraints	Lagrange Multipliers
t_{dye}	15 min	2.31	g_1	λ_6=0.868
f_{dye}	1/4		g6	λ_7=0.448
V_{dye}	29.6 mL		g14	λ_8=0.0468
l_{braid}	15.25 cm		g16	
D_{braid}	1.26 cm		g18	
f_{braid}	1/52			
f_{th}	1/2 weeks			
T_{th}	475 K			

Table 3.4.2 MATLAB "fmincon" Results for Starting Point $x_0=(1,1,1,1,1,1)^T$

3.6 Discussion of Results

A simple parametric study was done where the value for C2s was varied. It was one of the parameters involved with one of the nonlinear constraints identified previously (g1). When this parameter is increased, the values for the objective decreased. However, when decreased below the value of 0.4, no feasible solution could be found.

The results showed tradeoffs existed between the design variables l_{braid} and D_{braid}. However, due to limitations of the model, the values obtained mostly hit the bounds. The values for the dye model were acceptable in that they involved chemical processing. The results made physical sense, although implementing the values together may not necessarily achieve a beautiful style as typically seen.

4. HAIR DAMAGE (HAIR HEALTH)

4.1 Problem Statement

There are three general types of human hair which can be associated with the three major groups of the human race, i.e., Caucasian, Negroid and Oriental. Among these African-American hair can be associated with the Negroid type and is distinctive because of a high frequency of kinks along the fiber axis. This unique fiber configuration leads to extensive entanglements which present difficulties in combing and other grooming procedures. As a result grooming practices have been developed that can cause considerable damage to the hair.

One such practice is hot combing with metal comb, during which the hair is stretched at relatively high temperatures. This can cause partial straightening and possible damage to the surface of the fiber. Extensive loss of cuticle is frequently observed in African-American hair which has been subjected to this type of treatment, resulting in poor mechanical behavior. Another such practice is chemical straightening using alkaline chemicals (relaxers) which break both the disulphide and protein chains in the hair strand. As it hydrolyses protein chains which build up the backbone of the hair fibers in an irreversible manner, it can cause strong damage to hair.

Also, repeated combing or picking of a hair assembly is equivalent to subjecting the fiber to cyclic tensile loading, or tensile fatigue, which can have damaging effects on fiber structure. Because of the twisting of the African-American hair fiber over its longitudinal axis, such tensile loading will also develop torsion stresses, thus subjecting the fiber to both tensile and torsion fatigue. Generally African-American hair has a lower extension to break than Caucasian hair. It is also common observation that African-American hair, when subjected to mechanical handing, break into half-wave-length sections. This observation suggests that the regions of twist along the fiber are particularly prone to damage, so that failure occurs at relatively low levels of extension.

Thus, it is evident that damage would be caused to the hair if it needs to be groomed and more so in the case of African-American hair. But, the need would be to minimize the damage caused due to all the grooming and styling operations that have to be performed in order to have a presentable appearance. Thus, there exists a definite trade-off between having strong, healthy, undamaged hair and straightened and styled hair.

4.2 Nomenclature

The following symbols are used for the variables, parameters, and constants involved in this optimization problem:

Symbols	Meaning	Units
σ_0	Original fracture stress of hair	MPa
σ_{frac}	Fracture strength of hair	MPa
$T_{thermal}$	Temperature during thermal relaxing	K
$f_{thermal}$	Frequency of thermal relaxing	#applications/week
τ_{braid}	Twist factor	-
f_{braid}	Frequency of braiding wear	#of times/month
F_{comb}	Combing force	N
f_{comb}	Frequency of combing	#of times/day
t_r	Duration time of chemical application	S
f_r	Frequency of relaxer application	#applications/week
μ	Coefficient of friction of hair	-
R_i	Reaction force during combing	N/hair
n_i	Number of hair in contact	-
RH	Relative Humidity	-
A_{braid}	Size of braid	# of turns/cm
α_{braid}	Angle of twist	radians
$\Delta t_{chemical}$	Lag time	min
$\tau_{chemical}$	Characteristic straightening time	S
$C_{chemical}$	Chemical residual stress quotient	months

$C_{thermal}$	Thermal residual stress quotient	months
C_{braid}	Braiding residual stress quotient	months
C_{comb}	Combing residual stress quotient	days
A_{hair}	Cross-sectional area of hair	μm^2
$K_{chemical}$	Chemical elongation to fracture stress proportionality constant	MPa
ρ_{hair}	Hair density	#hairs/ cm^2
a	Major Ellipse diameter	μm
b	Minor Ellipse diameter	μm
dl/dt	Hair growth rate	μm/day
l_{old}	coiled length of hair before treatment	Cm
l_{new}	coiled length of hair after treatment	Cm
L	Length of Hair	Cm

Table 4.2.1 Hair Damage Subsystem Nomenclature

4.3 Mathematical Model

4.3.1 Objective Function

The primary objective is to minimize the breakage of hair caused by the grooming and styling operation. A hair strand would break if it subjected to a stress beyond its fracture stress value. Thus, in order to minimize the probability of failure, the strength of the hair needs to be maximized. This implies that the mathematical objective would be to maximize fracture stress thereby reducing the probability of failure for the hair.

Fracture stress is affected by the thermal, chemical and mechanical treatment that the hair strand is subjected to.

Effect of thermal treatment

It has been observed that during thermal treatment there is no correlation between the straightening effect produced (Ratio of length after to length before the treatment) and the mechanical tension applied. Though, possibly, there is a minimum tension needed to break the hydrogen bonds which exists between 0 N to 0.0089 N (Read, 2004). Thus, for our model, we have neglected the effect of this tension that is applied while the hair is being treated.

The effect of the temperature on the yield stress of the hair has been extrapolated from the stress-strain curves for different temperature in the range of 295 to 320 K. The yield stress decreases approximately linearly with temperature with a slope equal to 0.17 $MPaK^{-1}$. As the post yield stress value remains almost constant, the same gradient relation can be assumed for the fracture stress too (Akkermans, 2004).

$$\sigma_{frac} = \sigma_0 - 0.17T_{thermal} \qquad \text{Equation 4.3.1}$$

Effect of mechanical treatment

The effect of mechanical treatment is split into two categories-

1. Effect of Braiding
2. Effect of combing

Braiding – Braiding generates torsion in the hair fiber due to twisting. This twisting of the hair fiber increases the strain on the fiber structure, eventually locking the structure as an angel to the fiber, and causing molecular slippage, resulting in broken bonds and hence failure of the fiber. The data for the "twist and break experiment" was used to find the relationship of the twist factor (τ) and the fracture stress (Dankovich, 2004).

$$\sigma_{frac} = \sigma_0 + 0.43\tau_{braid} - 0.013\tau^2_{braid} \qquad \text{Equation 4.3.2}$$

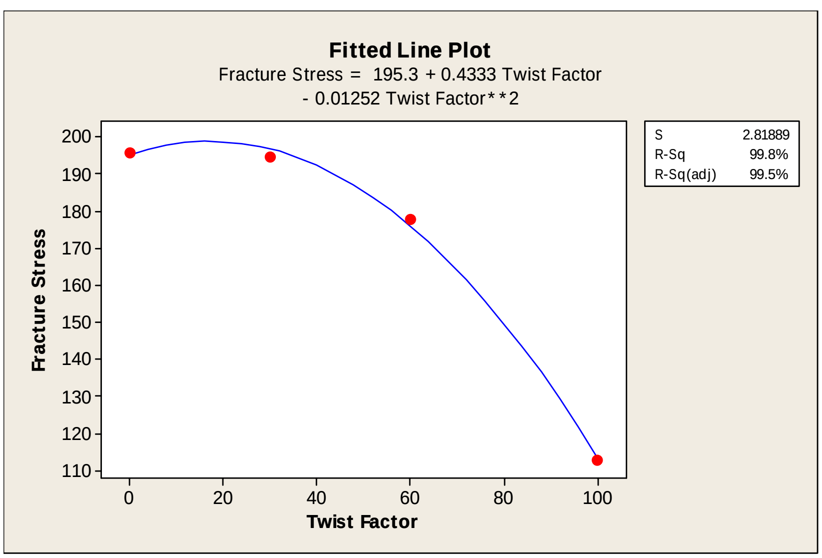

Fig 4.3.1 Relation between Twist Factor and Fracture Stress

Combing – Combing forces generate axial load on the hair which induces a temporary stress in the hair fiber. The effect of this stress reduces with time and thus we assume that the effect of combing on the fracture stress would be based on the residual stress retained by the hair fiber after combing.

$$\sigma_{frac} = \sigma_0 - \frac{F_{comb}}{A_{hair}}\left(e^{-\frac{C_{comb}}{f_{comb}}} \right) \quad \text{Equation 4.3.3}$$

<u>Effect of chemical treatment</u>

A chemical treatment with chemicals like dyes and relaxers makes the hair weak by breaking certain chemical bonds. In particular, relaxers have a straightening effect on the hair which increases with the duration for which the chemical is applied. Thus, this increase in length can be considered as an indication of the number of bonds broken and hence the amount by which the strength of the hair is reduced.

$$\sigma_{frac} = \sigma_0 - K_{chemical}\left(1 - e^{-\left(\frac{t_r - \Delta t_{chemical}}{\tau_{chemical}}\right)} \right) \quad \text{Equation 4.3.4}$$

Thus, the overall objective function would be:

$$\max.\ \sigma_{frac} = \sigma_0 - 0.17T_{thermal}\left(1 + e^{-\frac{C_{thermal}}{f_{thermal}}} \right) + \left(0.43\tau_{braid} - 0.013\tau^2{}_{braid} \right)\left(1 + e^{-\frac{C_{braid}}{f_{braid}}} \right)$$
$$- \frac{F_{comb}}{A_{hair}}\left(e^{-\frac{C_{comb}}{f_{comb}}} \right) - K_{chemical}\left(1 - e^{-\left(\frac{t_r - \Delta t_{chemical}}{\tau_{chemical}}\right)} \right)\left(1 + e^{-\frac{C_{chemical}}{f_r}} \right)$$

Equation 4.3.5

This can be modified and represented in the negative null form as:

$$\min.\ f(x;p) = \sigma_0 - \sigma_{frac} = 0.17T_{thermal}\left(1 + e^{-\frac{C_{thermal}}{f_{thermal}}} \right) - \left(0.43\tau_{braid} - 0.013\tau^2{}_{braid} \right)\left(1 + e^{-\frac{C_{braid}}{f_{braid}}} \right)$$
$$+ \frac{F_{comb}}{A_{hair}}\left(e^{-\frac{C_{comb}}{f_{comb}}} \right) + K_{chemical}\left(1 - e^{-\left(\frac{t_r - \Delta t_{chemical}}{\tau_{chemical}}\right)} \right)\left(1 + e^{-\frac{C_{chemical}}{f_r}} \right)$$

Equation 4.3.6

4.3.2 Constraints

The first constraint of the system would be based on the need to comb the hair at a minimum desired frequency which would be a function of the hair properties like hair density, curl diameter etc. such that the hair does not get tangled. In general, one should comb their hair at least twice a day.

$$g_1: 2 - f_{comb} \leq 0 \qquad \text{Equation 4.3.7}$$

The second constraint related to combing forces would be a bound on the amount of force that has to be used while combing. This is the minimum amount of force that will be needed to brush the hair.

$$g_2: \frac{\sum n_i R_i \mu}{\sum n_i} - F_{comb} \leq 0 \qquad \text{Equation 4.3.8}$$

The third constraint relates the coefficient of friction (μ) for the hair to the Relative Humidity (Moisture content of the hair)

$$g_3: 5\times 10^{-6} - 0.095(RH) - \mu \leq 0 \qquad \text{Equation 4.3.9}$$

The twist factor is related to the braid size (turns/cm) and the twist angle, by the following equalities.

$$h_1: \tau_{braid} = 1380\pi ab \times (A_{braid}) \qquad \text{Equation 4.3.10}$$

$$h_2: \tan\alpha_{braid} = \frac{0.0112\tau_{braid}}{1.175} \qquad \text{Equation 4.3.11}$$

The bound for A_{braid} will be an input from the other sections depending on styling requirement. For this model, the size of the braids is assumed to be governed by the following two constraints.

$$g_5: 0.5 - A_{braid} \leq 0 \qquad \text{Equation 4.3.12}$$

The frequency for braiding, thermal and chemical treatments is governed by the growth rate of the hair. The growth rate is a parameter and its value is assumed to be 260μm/day.

g_6: $0.33 - f_{braid} \le 0$ Equation 4.3.13

g_7: $0.33 - f_{thermal} \le 0$ Equation 4.3.14

g_8: $0.33 - f_r \le 0$ Equation 4.3.15

But keeping the hair braided (Duration of braid) affects the frequency of combing and thermal treatment. We assume that a person likes to keep his hair braided for 50% of the time between next braiding.

g_9: $-0.066f_{braid} + f_{comb} \le 0$ Equation 4.3.16

g_{10}: $-2f_{braid} + f_{thermal} \le 0$ Equation 4.3.17

To get the desired change in length, the chemical needs to be applied for certain duration of time. The relation between elongation and duration is as below. The value for the lag and characteristic straightening time are dependent on the chemical being used. For this case we have assumed the chemical to be NaOH and hence their corresponding values are, $\Delta t_{chemical}$ = 1.6 min and $\tau_{chemical}$ = 14 min.

$$g_{12}: \frac{l_{old} - l_{new}}{L_{total} - l_{old}} - \left(1 - e^{-\left(\frac{t_r - \Delta t_{chemical}}{\tau_{chemical}}\right)}\right) \le 0 \qquad \text{Equation 4.3.18}$$

4.4 Model Analysis

4.4.1 Well-boundedness check

	T_{th}	f_{th}	A_{braid}	f_{braid}	F_{comb}	f_{comb}	μ	RH	t_r	f_r
f	+	+	U	+	+	+			+	+
g1						-				
g2					-		+			
g3							-	-		
g5			-							
g6				-						
g7		-								
g8										-
g9				+		-				
g10		-		+						
g12									-	

Table 4.4.1 Preliminary Monotonicity Table

The preliminary monotonicity analysis indicated that the model is not well-bounded because of the following reasons:

1. There is no lower bound for $T_{thermal}$.
2. As the variable "RH" is not present in the objective, the current situation would make g_3 redundant which would next imply that g_2 is redundant too and hence F_{comb} will not have a bound at all.
3. Even though A_{braid} does not require an upper bound based on Monotonicity, but we need to include a practical upper bound just to make sure that the minimizer lies in the practically feasible space.

Further, some of the variables were observed to be hitting their bounds:

1. The constraints g_6 and g_8, are active and hence can be treated as equalities, thus giving a definite value to f_{braid} and f_r. Further, for f_{comb} and f_{th}, we have to evaluate which is the dominant constraint and hence they'll hit bounds too. There needs to be some more relationship that shall govern these frequencies.
2. g_{12} is active with respect to t_r and hence can be reduced for t_r too.

The set of constraints were thus modified by adding certain new constraints and deleting some old ones to ensure that the objective is well-bound:

g_1: $393 - T_{thermal} \leq 0$

g_2: $0.33 - f_{thermal} \leq 0$

g_3: $0.5 - A_{braid} \leq 0$

g_4: $-5 + A_{braid} \leq 0$

g_5: $0.33 - f_{braid} \leq 0$

g_6: $-30 + f_{braid} \leq 0$

g_7: $-2f_{braid} + f_{thermal} \leq 0$

g_8: $2 - f_{comb} \leq 0$

g_9: $1 - F_{comb} \leq 0$

g_{10}: $0.33 - f_r \leq 0$

g_{11}: $-2f_{braid} + f_r \leq 0$

g_{12}: $15 - t_r \leq 0$

h_1: $\tau_{braid} = 1380\pi ab \times (A_{braid})$

	T_{th}	f_{th}	A_{braid}	f_{braid}	F_{comb}	f_{comb}	t_r	f_r
f	+	+	U	+	+	+	+	+
g1	-							
g2		-						
g3			-					
g4			+					
g5				-				
g6				+				
g7		+		-				
g8						-		
g9					-			
g10								-
g11				-				+
g12							-	

Table 4.4.2 Modified Monotonicity Table

An important observation was made when the model with the above constraints was run. It was expected that all the variables (except A_{braid}) will hit their respective lower bounds, but that did not happen. Even though the function was expected to be decreasing with respect to f_{braid}, it was observed that f_{braid} was being maximized. This was because of the fact that, at the optimum, the contribution from the term with τ_{braid} was negative. This term formed the co-efficient for the term with f_{braid} and thus the objective was being maximized by increasing f_{braid}.

$$\left(0.43\tau_{braid} - 0.013\tau^2{}_{braid}\right)\left(1 + e^{-\frac{C_{braid}}{f_{braid}}}\right)$$

Thus, the monotonicity of the objective with respect to f_{braid} depends on the co-efficient term. As this co-efficient term changes sign in the feasible range (goes from negative to positive), the monotonicity with respect to f_{braid} is no longer definitive. Thus, the monotonicity was changed to "R" for f_{braid} as it is regionally monotonic depending on the values for τ. What this indicated is that f_{braid} needs an upper and a lower bound so as to ensure that the model is well bounded.

The physical interpretation of this situation is that braiding actually strengthens the hair (by making it less subject to stress from combing and thermal treatments) till a certain threshold twist value and if the hair is twisted further, it starts to weaken. Thus if we braid the hair for that optimum twist, it is beneficial for the hair and hence is advisable to be done as frequently as possible.

4.4.2 Non-linear Trade-offs

At this point it can be observed that the subsystem is well-bounded and satisfies the necessary condition to have a result but it does not have any non-linear design trade-offs between the variables. The solution can be obtained by solving for the minimum for each of the variables independently and 7 out of 8 variables are hitting their bounds.

The nature of the system indicates that certain trade-offs exist between the variables but there is no mathematical formulation for the same. It is observed that increasing the temperature during thermal treatment helps in straightening the hair and thus reduces the curl angle. This would imply that there would be reduction in combing forces required to comb the hair. The following mathematical formulation was assumed to represent the relationship between temperature and force. Though the two variables are actually linked by the curl angle, for simplicity this relation has been used. Similarly, as the relaxer too has a straightening effect, the time for which it is applied can be related to F_{comb} in a similar fashion.

g_{13}: $518.9 - 247.7F_{comb} + 119.5F^2{}_{comb} - T_{thermal} \leq 0$

g_{14}: $34.65 - 53.5F_{comb} + 36.46F^2{}_{comb} - t_r \leq 0$

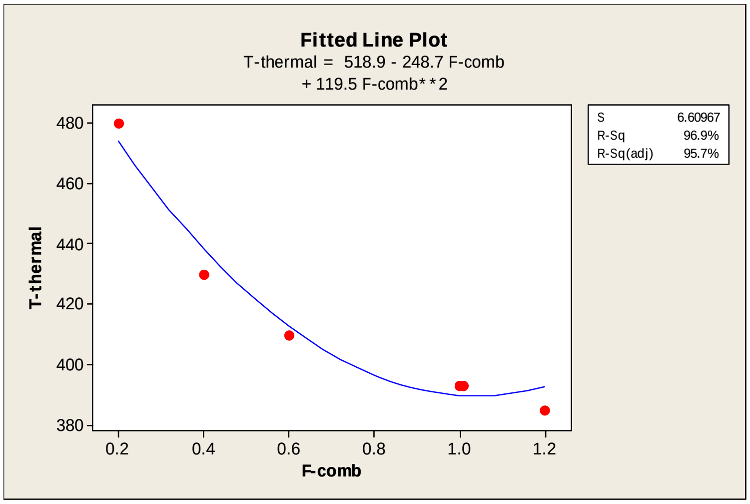

Fig 4.4.1 Relation between Temp. and Comb force

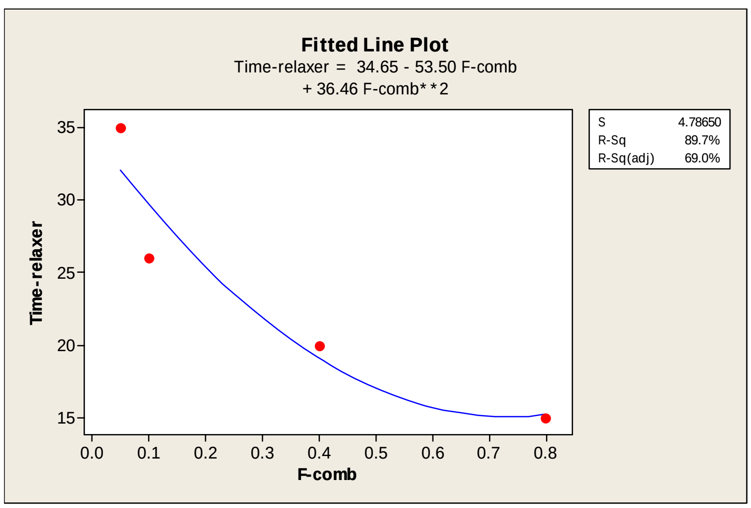

Fig 4.4.2 Relation between duration of chemical treatment and Comb force

Thus the complete model can now be summarized as below :

min. $f(x;p) = \sigma_0 - \sigma_{frac} = \quad 0.17T_{thermal}\left(1+e^{-\frac{C_{thermal}}{f_{thermal}}}\right) - \left(0.43\tau_{braid} - 0.013\tau^2{}_{braid}\right)\left(1+e^{-\frac{C_{braid}}{f_{braid}}}\right)$

$$+\frac{F_{comb}}{A_{hair}}\left(e^{-\frac{C_{comb}}{f_{comb}}}\right) + K_{chemical}\left(1-e^{-\left(\frac{t_r-\Delta t_{chemical}}{\tau_{chemical}}\right)}\right)\left(1+e^{-\frac{C_{chemical}}{f_r}}\right)$$

subject to:

g_1: 393 - $T_{thermal} \leq 0$

g_2: 0.33 – $f_{thermal} \leq 0$

g_3: 0.5 - $A_{braid} \leq 0$

g_4: -5 + $A_{braid} \leq 0$

g_5: 0.33 - $f_{braid} \leq 0$

g_6: -30 + $f_{braid} \leq 0$

g_7: -2f_{braid} + $f_{thermal} \leq 0$

g_8: 2 - $f_{comb} \leq 0$

g_9: 1 - $F_{comb} \leq 0$

g_{10}: 0.33– $f_r \leq 0$

g_{11}: -2f_{braid} + $f_r \leq 0$

g_{12}: 15 - $t_r \leq 0$

g_{13}: $518.9 - 247.7F_{comb} + 119.5F^2{}_{comb} - T_{thermal} \leq 0$

g_{14}: $34.65 - 53.5F_{comb} + 36.46F^2{}_{comb} - t_r \leq 0$

h_1: $\tau_{braid} = 1380\pi ab \times (A_{braid})$

Thus, the complete model monotonicity table is as shown below:

	T_{th}	f_{th}	A_{braid}	f_{braid}	F_{comb}	f_{comb}	t_r	f_r
f	+	+	U	R	+	+	+	+
g1	-							
g2		-						
g3			-					
g4			+					
g5				-				
g6				+				
g7		+		-				

g8						-		
g9					-			
g10								-
g11				-				+
g12							-	
g13	-				U			
g14					U		-	

Table 4.4.3 Modified Monotonicity Table

Using MP1 the following activities are identified:

- ✓ $\{g_1,g_{13}\}$ is active with respect to $T_{thermal}$.
- ✓ $\{g_2\}$ is active with respect to $f_{thermal}$
- ✓ $\{g_8\}$ is active with respect to f_{comb}
- ✓ $\{g_{10}\}$ is active with respect to f_r
- ✓ $\{g_{12},g_{14}\}$ is active with respect to t_r

To identify which of $\{g_1,g_{13}\}$ is active, the constraint space was plotted as shown in fig 4.4.3

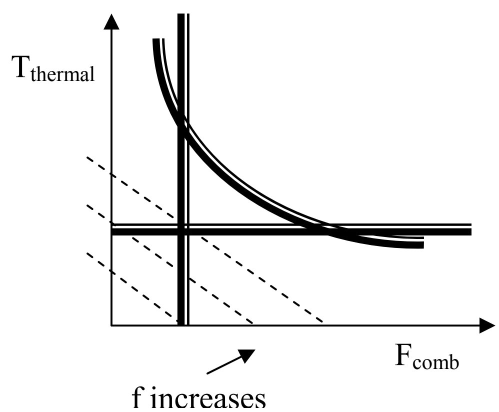

Fig 4.4.3 Graphical representation of constrained space

Analytical results indicate that g_{13} is active. A similar analysis for $g_{12}$and g_{14}indicates that g_{14} is active.

We are left with 3 degree's of freedom as there are 5 active constraints for the 8 variables in the subsystem. Of these 5, 3 are simple bounds, they can be analytically solved to get the following results:

1. $f_{thermal}$ = 0.33 times/month
2. f_{comb}= 2 times/day
3. f_r = 0.33 times/month

As the co-efficient of f_{braid} depends on A_{braid}, its monotonicity is not very evident from the monotonicity table. But, as A_{braid} is an independent variable, the objective is minimized if we find the minimizer A_{braid} in the constrained space. This results in an interior solution which makes the co-efficient for f_{braid} negative and hence, g_6 is active. Therefore, we can add onto the results list above:

4. f_{braid} = 30 times/month

Thus, we are now left with only 2 degree's of freedom. Substituting the active constraints g13 and g14 makes the objective very complex and difficult to solve analytically. Numerical Methods had to be used to calculate the results.

4.5 Optimization Study

As described in the previous section, the values for 4 variables were obtained based on the active bounds. As A_{braid} was identified as an independent variable, the function was minimized with respect to A_{braid} within the constrained space and it was observed to have a interior minima. The value was calculated by equating the gradient to zero, and then checking for the Hessian value at that point. It was found that the Hessian was positive definite and hence the following values for 5 of the variables was obtained.

- $f_{thermal}$ = 0.33 times/month
- f_{comb}= 2 times/day
- f_r = 0.33 times/month
- f_{braid} = 30 times/month
- A_{braid} = 0.7078 turns/cm

To calculate the values for the remaining three variables, the Matlab optimizer fmincon was used. An 8 degree of freedom problem was programmed and the constraint activities were confirmed to be as identified by the monotonicity analysis.

	x*	f*	Active Constraints	Lagrange Multipliers
T-thermal	401.76	132.83	g2	0.2120
f-thermal	0.33		g6	0.0107
Twist factor-braid	16.53845		g8	7.2161
f-braid	30		g10	0.1914
F-comb	0.730		g13	0.1700
f-comb	2		g14	2.7383
tr	15.025			
fr	0.33			
A-braid	0.708			

Table 4.5.1 MATLAB "fmincon" Results for Starting Point $x_0=(0,0,0,0,0,0,0,0,0)^T$ or $x_0=(100,100,100,100,100,100,100,100,100)^T$

The results agree with the analytical solution and give results as expected. Even different starting points converge to the same result thus indicating global convergence of the model.The lambda values indicate the sensitivity of the objective to each of the constraints. The lambda values for the 6 active constraints is less than 1 for 4 out of the 6 constraints and hence indicates that the objective is relatively insensitive to changes in the constraints thus indicating stability of the model. The two constraints that have larger values have lambdas of the order of 2 and 7 which is not very high either. But, the fact that the highest sensitivity is with respect to a lower bound on frequency of combing indicates that changing this limit might have the most significant effect on the overall objective.

In order to understand the trade-offs better, the program was run under different combinations of the constraints:

1. Excluding both g_{13} and g_{14} – This resulted in all three bounds for F_{comb}, $T_{thermal}$ and t_r to be active (as expected)
    a. F_{comb} = 0.1 N
    b. $T_{thermal}$ = 393 K
    c. t_r =15 min

2. Including g_{13} only – This resulted in a trade off between $T_{thermal}$ and F_{comb} and the minimizer value shifted to a point on g_{13} thus making it active. This point was confirmed analytically by considering the tangent to constraint having the slope equal to the objective function. Both numerical and analytical results agreed.
    a. $F_{comb} = 0.702$ N
    b. $T_{thermal} = 403.8$ K
    c. $t_r = 15$ min.

3. Including g_{14} only – Similar to the previous case, this resulted in a trade-off between F_{comb} and t_r and g_{14} became active. An exact analytical solution could not be calculated due to the numerical complexity of the problem, but the behavior was similar to that of the previous case.
    a. $F_{comb} = 0.66$ N
    b. $T_{thermal} = 393$ K
    c. $t_r = 15.23$ min.

4. Including both g_{13} and g_{14} – As the optimum for g_{13} and g_{14} had different values for the common variable F_{comb}, when both of them were introduced, both were active and the resulting value was a trade-ff between the two minimizer points obtained from the previous two cases.
    a. $F_{comb} = 0.73$ N
    b. $T_{thermal} = 401.7$ K
    c. $t_r = 15.025$ min.

These results improve the confidence in the behavior of the overall subsystem model and its stability.

An interesting observation was made with respect to A_{braid} and f_{braid}. The current minimizer value is an interior solution, but if the lower bound for A_{braid} is increased such that the minimum lies outside the feasible domain, it becomes active. Further shifting the bound higher, has an effect on the behavior of f_{braid}, and after a cutoff at 1.4 turns/cm, the lower bound for f_{braid} becomes active.

This behavior can be explained based of the fact that the co-efficient for f_{braid} becomes positive in such a case. The physical interpretation of this result is that braiding is "good" and actually provides strength to the hair when they are braided but only to a certain twist. Increasing the twist angle further starts to induce torsional damage to the hair and hence making it weaker. Thus, as long as the twist factor does not cause damage to the hair, braiding should be done as frequently as possible, but if the braiding causes damage, its frequency should be minimized

4.6 Parametric Study

A parametric study with respect to the parameters $C_{thermal}$, C_{braid}, C_{comb} and $C_{chemical}$ was conducted and an observation was made that once these parameters were increased beyond 10, the model started to give different results based on the starting point selected to start the optimization. This was due to the fact that these parameters are a part of an exponential term which tends to zero if these parameters are large. As the actual range for these parameters is unknown, this leads us to define a model validity constraint, limiting the use of the model with these parameters taking a maximum value of 5. An important thing to note though is that even though the minimizer value is affected the overall objective value remains the same even if these parameters are varied beyond the range.

4.7 Discussion of Results

The results obtained from the subsystem are realistic and can be correlated to the general thumb rules followed for hair care. It indicates that braiding is the best for hair and should be exercised as often as possible. But, it also indicates that getting tight braids is not advisable as they tend to cause damage due to torsion. Thus the braid size should be limited to 1.4 turns/cm but for maximum results 0.7 turns/cm is suggested. It also reflects that manageability and combing becomes easier if the hair is straightened using thermal treatment and chemical treatment. An important thing to keep in mind is the sensitivity of the model to frequency of combing as the limit on frequency of combing has been derived out of usual daily practices. The results correlate well with the common practice and advice by the hair-dressers.

Further, when the subsystem interacts with the overall system, it is expected represent practical trade-offs based on the fact that in order to maximize hair length and styling, thermal and chemical treatment would have to be increased which would cause damage. It should though indicate that braiding is a preferred styling option as it causes least damage (assuming optimum size).

Due to the limitations there were quite a few aspects of hair care that were ignored one of the most important being the effect of moisture content of the hair. In addition to thermal and chemical relaxers, the moisture content is affected by the application of various chemicals like oil and water too. This makes it quite difficult to consider the moisture content. Even the chemicals being used have a wide range and hence only one of them was selected. The chemical composition and the effect it has on the straightening was simplified and assumed to be a function of time alone without considering the effect of strength of the chemical. Also, the environment plays an important role which was ignored.

5. SYSTEM OPTIMIZATION: HAIR BEAUTY

5.1 Problem Statement

As described in the introduction, many African-American women desire to have hair that is straightened from its natural, curly state such that their hair appears to be longer. This is important to them because straighter, longer hair often enhances the capability to style and manage their hair. Another key trend is that black women want to have the freedom to style their hair anyway they choose. Above all, these women wish to have hair that can undergo these first two processes and remain perfectly healthy. However, this last trend is in direct conflict with the other two because many of the straightening and styling techniques used on black women's hair are inherently damaging to the hair. Hence, these tradeoffs among hair straightening, styling, and health provide the motivation for a system-wide study to maximize the beauty of African-American women's hair.

5.2 Mathematical Model

Because the system model is somewhat simple from an optimization perspective (i.e., not a high degree of coupling), an All-in-One (AIO) approach can be used to formulate and solve the problem. Therefore, all variables, parameters, and constants included in this problem can be referenced in their respective subsections. The goal of the system model is to maximize hair beauty of an African-American woman subject to some physical and practical limitations. Letting F denote hair beauty and f_1, f_2, f_3, denote the hair length, hair damage, and hair styling functions respectively, we arrive at the following objective function to be minimized:

$$-F = \frac{f_1}{\Delta L_{eff,\max}} + \frac{f_2}{\sigma_0} + \frac{f_3}{3} \qquad \text{Equation 5.2.1}$$

It should be observed that this equation is in negative null form and that its output F is dimensionless. To ensure this property, the functions f_1 and f_2 are normalized by their maximum possible values $\Delta L_{eff,max}$ and σ_0, respectively. The function f_3, although already dimensionless, is normalized by its maximum possible value of 3 in order to ensure that each sub-function produces values that are on the same order of magnitude. Physically, the overall function output can be thought of as an "index" number indicating the degree of beauty of an African-American woman's hair. Hence, in negative null form, as the output becomes more negative, the hair can be described as more "beautiful". Any positive number is undesirable since this would indicate that the damage function dominates the length and styling functions and that essentially limited or no hair treatments are possible.

Including the constraints from all of the subsystems, the overall system problem can be summarized as the following:

$$\min\ -F = \frac{f_1}{\Delta L_{eff,\max}} + \frac{f_2}{\sigma_0} + \frac{f_3}{3}$$

subject to $g_1 = C_4 m_r f_r - C_{relax} \leq 0$

$$g_2 = C_4 t_r f_r - 0.0002484 \leq 0$$

$$g_3 = C_4 t_{th} f_{th} - 0.000011574 \leq 0$$

$$g_4 = C_2 m_r T_{th} - 58.5 \leq 0$$

$$g_5 = -2 f_{braid} + f_{th} \leq 0$$

$$g_6 = -2 f_{braid} + f_r \leq 0$$

$$g_7 = 518.9 - 247.7 F_{comb} + 119.5 F^2{}_{comb} - T_{thermal} \leq 0$$

$$g_8 = 34.65 - 53.5 F_{comb} + 36.46 F^2{}_{comb} - t_r \leq 0$$

$$g_9 = D_{braid} l_{braid} - 24.06 C_{2s} \leq 0$$

$$g_{10} = T_{th} f_{th} - 3.928 \times 10^{-6} C_{6s} \leq 0$$

$$g_{11} = 0.05 - m_r \le 0$$
$$g_{12} = m_r - 0.15 \le 0$$
$$g_{13} = 900 - t_r \le 0$$
$$g_{14} = t_r - 1200 \le 0$$
$$g_{15} = 1/8 - f_r \le 0$$
$$g_{16} = f_r - 1/4 \le 0$$
$$g_{17} = 393 - T_{th} \le 0$$
$$g_{18} = T_{th} - 475 \le 0$$
$$g_{19} = 1 - t_{th} \le 0$$
$$g_{20} = t_{th} - 3 \le 0$$
$$g_{21} = 1/2 - f_{th} \le 0$$
$$g_{22} = f_{th} - 7 \le 0$$
$$g_{23} = 11.6871 - \tau \le 0$$
$$g_{24} = \tau - 116.817 \le 0$$
$$g_{25} = 0.33 - f_{braid} \le 0$$
$$g_{26} = f_{braid} - 4 \le 0$$
$$g_{27} = 0.1 - F_{comb} \le 0$$
$$g_{28} = F_{comb} - 3 \le 0$$
$$g_{29} = 2 - f_{comb} \le 0$$
$$g_{30} = f_{comb} - 10 \le 0$$
$$g_{31} = 0.5 - A_{braid} \le 0$$
$$g_{32} = A_{braid} - 5 \le 0$$
$$g_{33} = 15 - t_{dye} \le 0$$
$$g_{34} = t_{dye} - 30 \le 0$$
$$g_{35} = 0.09123 - f_{dye} \le 0$$
$$g_{36} = f_{dye} - 0.5 \le 0$$
$$g_{37} = 29.6 - V_{dye} \le 0$$
$$g_{38} = V_{dye} - 118.3 \le 0$$
$$g_{39} = 15.24 - l_{braid} \le 0$$
$$g_{40} = l_{braid} - 60.96 \le 0$$
$$g_{41} = 0.3175 - D_{braid} \le 0$$
$$g_{42} = D_{braid} - 1.5785 \le 0$$

$$h_1 = \frac{2\pi Lk\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\dfrac{0.5d_{app}}{0.5d_{app} - 0.5x10^{-6}}\right)} - \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 Lc_v (T_{bp} - T_{room})}{t_{th}} = 0$$

$$h_2 = \tau - 1380 A_{hair} A_{braid}$$

$$h_3 = D_{braid} + 0.2802 A_{braid} - 1.7186$$

Note that the system model has 45 constraints and 16 variables, 7 of which are linking variables. These terms include t_r, f_r, T_{th}, f_{th}, f_{braid}, A_{braid} and D_{braid}.

5.3 Model Analysis/Optimization Study

The traditional approach of using monotonicity analysis prior to optimization has been omitted due to the complexity (size) of the AIO problem and due to several evolutions of the subsystem models being input to the system model. Therefore, the system model that is described above has been directly entered into a numerical optimization tool, with the intent of verifying its validity through examining outputs of KKT conditions. Initially, a code has been written in MATLAB that makes use of its "fmincon" function (Appendix B.X) to find the optimal solution. Observe that no scaling has been used despite the differences in the orders of magnitudes of the variables. This has been done intentionally to determine the efficacy and robustness of the subsystem model in a direct way. Additionally, the parameter C_4 has been changed to 1.25 for the system analysis because its nominal value of 1.5 produces infeasible solutions in the MATLAB solver. Since the values for this parameter can range from 1-2 (see 2.6 Parametric Study), 1.25 is a reasonable value. Finally, note that the constraint tolerances have been set to $1x10^{-10}$ to generate higher accuracy in the algorithm. The key items that are reported in the output include the optimal point (x*), the function value at the optimal point (f*), constraint activity, and the Lagrange multiplier values. This information is highlighted in Tables 5.3.1 for an initial starting point of the lower bounds. As a final item, an "exit flag 4" has been reported from the algorithm. This indicates that the solver terminated because the magnitude of the vector in the search direction is insufficient, i.e., its value is too small.

	x*	f*	Active Constraints	Lagrange Multipliers
m_r	7.37E-02	-3.13E-01	h_1	1.19E-02
t_r	9.00E+02		h_2	-4.84E-04
f_r	2.21E-07		h_3	-3.90E-02
T_{th}	3.97E+02		g_2	4.06E+02
t_{th}	2.58E+00		g_4	1.87E-03
f_{th}	8.27E-07		g_7	9.58E-04
τ	7.75E+01		g_9	1.26E-02
f_{braid}	4.00E+00		g_{13}	1.71E-04
F_{comb}	8.10E-01		g_{21}	1.09E+05
f_{comb}	2.00E+00		g_{26}	1.45E-02
A_{braid}	3.32E+00		g_{29}	3.15E-02
t_{dye}	3.00E+01		g_{34}	5.84E-04
f_{dye}	5.00E-01		g_{36}	3.50E-02
V_{dye}	1.18E+02		g_{38}	1.48E-04
l_{braid}	1.52E+01			
D_{braid}	7.89E-01			

Table 5.3.1 MATLAB "fmincon" Results for Starting Point x_0=Lower Bounds

Based on these results, the optimal solution appears to have two degrees of freedom, with seven of the variables hitting bounds. Additionally, the model appears to represent the expected trends of the physical process. However, in using other starting points for the algorithm, it has been found that global convergence for this solution is not definitive. An attempt at scaling has therefore been made (Appendix B.X) to alleviate this issue; nevertheless, this has produced results that are infeasible. This may be due to numerical function sensitivities that have been built into the model through "tweaking" parameters. The scaling issues may also be due the significant differences in the Lagrange multipliers (and hence sensitivities) for each constraint. Hence, another optimization tool, iSight, has been used to verify the MATLAB results. Note that for this solution process, a combination of a derivative-free algorithm (adaptive simulated annealing) and a gradient-based algorithm (nonlinear SQP) has been used to optimize this system. The model of the system itself has remained the same with the exception of the introduction of a slack variable for h_1, which is necessary to ensure the algorithm functions properly. These results are highlighted in Table 5.3.2.

	x^*	f^*	Active Constraints
m_r	7.15E-02	-2.59E-01	h_1
t_r	9.00E+02		h_2
f_r	2.21E-07		h_3
T_{th}	4.09E+02		g_2
t_{th}	2.30E+00		g_4
f_{th}	8.27E-07		g_7
τ	5.60E+01		g_9
f_{braid}	4.00E+00		g_{10}
F_{comb}	6.43E-01		g_{13}
f_{comb}	2.00E+00		g_{21}
A_{braid}	3.32E+00		g_{26}
t_{dye}	3.00E+01		g_{29}
f_{dye}	2.50E-01		g_{36}
V_{dye}	1.18E+02		
l_{braid}	1.52E+01		
D_{braid}	1.09E+00		

Table 5.3.2 iSight Results for Starting Point x_0=Lower Bounds

From these results, the optimal solution appears to have three degrees of freedom, with five of the variables hitting the bounds. Observe that activity for the constraints has been assumed when the value is within an accuracy of 10^{-5} of zero. The discrepancy between the iSight result (in terms of constraint activity) and that from MATLAB is most likely due to the inclusion of a derivative-free algorithm and a slack variable for h_1. Nevertheless, the results for the optimum and the optimizers for iSight are similar to those obtained from MATLAB. Because MATLAB uses a purely gradient-based algorithm (SQP) and its minimum is lower than that produced by iSight, the MATLAB result is considered the “true” optimum.

Although global convergence is not definitive, this model is still quite meaningful. First, the optimal value indicates a beauty index magnitude of 0.313, which, when compared to its maximum possible value of nearly 0.50, indicates a significant amount of “hair beauty”. The suggested chemical relaxing process involves about 0.073 kg of relaxer applied for 15 minutes (900 s) every 7 weeks. This is slightly more aggressive in terms of quantity (0.05 kg), but closer to conventional wisdom in terms of frequency (1 application/8 weeks) for the traditional chemical straightening process. The suggested thermal relaxing process involves about 397 K (124°C) of heat applied for just over 2.50 s (per strand(s) of hair, with a flat iron) between 1-2 times per week. Observe that this temperature and frequency schedule is also applicable for the thermal styling process. In terms of braiding, it is suggested that moderately-sized braids (~3 turns/cm) be worn once a week. This frequency is slightly more aggressive than what is seen in practice, which may be due to the strong influence that it has in preventing damage (see 4.7 Discussion of Results). Finally, it is suggested that about 118 mL of dye be applied for 30 minutes once each month. This is somewhat reasonable given the current trends in hair-dyeing for African-American women. Therefore, the design model that has been developed does in fact capture the essence of the complete hair care process with interesting tradeoffs.

6. ACKNOWLEDGMENTS

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Interviews:
Practical insights received through interviews with Cookie at Above Ground Hair Salon in Ann Arbor and with Rosaline at Rosaline's Beauty Salon in Ypsilanti.

APPENDICES

A. Previous Subsystem Work

A.1 Previous Hair Length Subsystem work

A.1.1 Nomenclature

The following symbols are used for the variables, parameters, and constants involved in this optimization problem:

Symbols	Meaning	Units
m_r	Quantity of relaxer (mass)	kg
t_r	Duration time of relaxer application	s
f_r	Frequency of relaxer application	#applications/week
t_{th}	Duration time of thermal relaxing	s
f_{th}	Frequency of thermal relaxing	#applications/week
T_{th}	Temperature applied during thermal relaxing	K
F_{th}	Force applied during thermal relaxing	N
N	Number of curls along effective hair length	None
d_{curl}	Curl diameter	m
σ_{frac}	Fracture stress of hair	MPa
$\Delta L_{eff,max}$	Maximum change in effective hair length	m
$L_{eff,0}$	Initial effective hair length	m
C_1	Exponential coefficient for curve diameter relation	m
C_2	Exponential coefficient for curve diameter relation	m
C_3	Exponential coefficient for curl relation	None
C_4	Exponential coefficient for curl relation	None
T_{room}	Ambient temperature during thermal relaxing	K
T_{bp}	Boiling point of water	K
k	Thermal conductivity of hair	W/m*K
c_v	Specific heat of water	J/kg*K
L	Actual hair length	m
d_{app}	Apparent circular diameter of hair	µm
A	Area of flat iron plate coverage	m^2
ρ_{hair}	Area density of hair	hairs/cm^2
ρ_{H20}	Mass density of water	kg/m^3
C_5	Chemical relaxer cost sensitivity coefficient	$*week/kg*application
C_{relax}	Cost of professional chemical relaxing	$

Table A.1.1 Effective Hair Length Subsystem Nomenclature

A.1.2 Mathematical Model

Objective Function

The objective function for this subsystem is to maximize the change in the effective hair length of an African-American woman with coarse hair texture through chemical and thermal relaxing. This would cause the overall effective hair length to increase, thus creating the appearance of longer hair. Note that the choice of an African-American woman with the hair texture described above has been made to consider a "worst case" system design scenario.

Prior to any formulations, the following major assumptions have been made:

5. The woman maintains a healthy diet and therefore does not negatively affect the hair growth rate.
6. The woman's scalp is healthy, i.e. there are no scrapes, cuts, or abrasions that might pose harm to her when exposed to harsh chemical relaxers.
7. The woman is not taking any drugs (medical or otherwise) that might negatively impact the hair growth rate.
8. Only one type/formulation of chemical relaxer (NaOH) is being used.

In deriving the objective function, exponential functions have been chosen to approximate the phenomena caused by the relaxing treatments. This is a good initial approach to modeling the problem for the following reasons:

4. Most phenomena in nature are described by nonlinear functions, and, in many cases, end up being expressed as a linear combination of exponentials when solved analytically via ordinary differential equation theory.
5. Similar expressions have been used in other effective length and hair straightening efficacy models (Quadlfieg, 2003).
6. The nature of the effects of the relaxing techniques is such that it causes the change in effective hair length to approach an asymptotic value, i.e., a woman will never be able to straighten her hair completely.

Using this framework, an exponential expression similar to that of a step input function has been generated describing the "positive" effects of chemical and thermal relaxing:

$$\frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th} F_{th}}\right) \qquad \text{Equation A.1.1}$$

Observe that in this expression, all variables are inputs into the argument of the exponential function for simplicity. Also, the general trends between the input variables and the change in effective hair length are captured: as each of the variables increase (more chemical or thermal relaxing), the change in the effective hair length increases sharply until it reaches a limiting value, which, when added to the initial effective hair length, will result in the actual hair length.

Similarly, the "negative" effects of the relaxing techniques are captured in another exponential expression. Here, it is desired to capture the fact that as the hair is subjected to more relaxing treatments, the fracture strength actually decreases over time. This decrease in fracture strength causes the hair to be more prone to breakage, which, by definition, decreases the effective hair length. By extrapolating from a plot in Akkermans, 2004, and incorporating the effects of chemical relaxing, the following relationship has been developed for fracture strength:

$$\sigma_{frac} = -.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r \qquad \text{Equation A.1.2}$$

Note that the coefficient for the first term in the above relation has been extrapolated from the slope of a linear plot and the coefficient of the second term is based on the assumption that the effects of chemical relaxing will be on the order of ten times more dominant than that of thermal relaxing. Using this formulation, the "negative" effects of the relaxing techniques can be described by:

$$-L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r}) \qquad \text{Equation A.1.3}$$

In this equation, note that when no relaxing is performed (i.e. the inputs are zero), there is no loss of hair length. However, as the amount of relaxing increases, more hair is lost, and, if performed without “bound”, could lead to complete hair loss. The total change in the effective hair length can then be described by combining Equations A.1.1 and A.1.3:

$$\Delta L_{eff} = \frac{\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th} F_{th}}\right) - L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r}) \quad \text{Equation A.1.4}$$

Finally, since it is desired to maximize the above objective function while solving it in a negative-null form, the equation is negated and appears in the following representation to be minimized:

$$-\Delta L_{eff} = \frac{-\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th} F_{th}}\right) + L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r}) \quad \text{Equation A.1.5}$$

Constraints

The first constraint for this subsystem is based off of mechanical spring theory, which states that the actual length of a spring is equivalent to the product of the circumference of a single coil and the number of coils comprising the spring. If an African-American female’s hair is modeled in a similar fashion, one can find the circumference of a single curl using the curl diameter and then find the actual length of hair by multiplying the circumference by the number of curls:

$$h_1 = \pi N d_{curl} - L = 0 \quad \text{Equation A.1.6}$$

This ensures that the curl diameter and the number of curls are varied such that they always equal the actual hair length.

The next constraint shows the necessary dependency of the curl diameter on the chemical and thermal relaxing variables. As with the objective function, exponential functions have been chosen to illustrate this relationship:

$$h_2 = d_{curl} - C_1 e^{m_r t_r f_r} - C_2 e^{T_{th} t_{th} f_{th} F_{th}} = 0 \quad \text{Equation A.1.7}$$

This shows the expected trend that as the input variables increase (more relaxing performed), the curl diameter increases sharply until it reaches "infinity", which would correspond to completely straight hair. In the trivial case that the input variables are set to zero, one would arrive at a value corresponding to the sum of the exponential coefficients. Physically, this would be the initial, or natural, curl diameter of a woman's hair. Note that similar to Equation A.1.2, these constants are constructed in such a way that the coefficient associated with chemical relaxing is approximately ten times more dominant than that of the thermal relaxing.

A similar constraint has also been developed for the number of curls based on the chemical and thermal relaxing input variables. This, too, shows the nature of the dependency of the number of curls on the input variables:

$$h_3 = N - C_3 e^{-m_r t_r f_r} - C_4 e^{-T_{th} t_{th} f_{th} F_{th}} = 0 \qquad \text{Equation A.1.8}$$

This shows the expected trend that as the input variables increase (more relaxing performed), the number of curls decreases sharply until it nearly reaches zero, which would correspond to completely straight hair. In the trivial case that the input variables are set to zero, one would arrive at a value corresponding to the sum of the exponential coefficients. Physically, this would be the initial number of curls in a woman's hair. Once again, these constants are constructed in such a way that the coefficient associated with chemical relaxing is approximately ten times more dominant than that of the thermal relaxing.

Another equality constraint for the subsystem shows the relationship between the fracture strength and the curl diameter (Porter, 2005). In general, as the curl diameter increases, the fracture strength increases due to increased strength in tension of the hair:

$$h_4 = \sigma_{frac} - 57 d_{curl} - 129 = 0 \qquad \text{Equation A.1.9}$$

The final equality constraint is based on energy balance during thermal relaxing. Since the only process being considered is the heating up of the hair, the (rate-based) energy equation reduces to

$$\dot{E}_{in} = \Delta\dot{E}_{internal}$$ Equation A.1.10

where the term on the left side of the equation describes all energy inputs into the system and the term on the right side of the equation describes the change in internal energy of the system. The only energy input into the system is conduction of the hair due to the flat iron, which, for a single strand of hair, is described by:

$$\dot{E}_{in} = \frac{2\pi Lk\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\frac{0.5d_{app}}{0.5d_{app} - 0.5x10^{-6}}\right)}$$ Equation A.1.11

Observe that this expression is based on conduction of a "hollow tube". The change in internal energy of the system is due to a temperature rise of the water within the hair, which is described by:

$$\Delta\dot{E}_{internal} = \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 Lc_v(T_{bp} - T_{room})}{t_{th}}$$ Equation A.1.12

Substituting Equations 2.3.11 and 2.3.12 into 2.3.10 provides the necessary energy balance constraint:

$$h_5 = \frac{2\pi Lk\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\frac{0.5d_{app}}{0.5d_{app} - 0.5x10^{-6}}\right)} - \frac{0.25\rho_{H20}\pi\left(d_{app} - 1x10^{-6}\right)^2 Lc_v(T_{bp} - T_{room})}{t_{th}} = 0$$

Equation A.1.13

The first inequality constraint limits the amount and frequency of chemical relaxing performed by cost. In general, it is known that these two input variables are inversely proportional to one another; if more chemical relaxer is applied during a given treatment, then the treatments must be made less frequently, and vice versa. An arbitrary sensitivity constant is added to the equation such that the product of this constant and the two variables will result in a total cost for professional chemical relaxing. This is expressed by:

$$g_1 = C_5 m_r f_r - C_{relax} \leq 0 \qquad \text{Equation A.1.14}$$

Another inequality constraint limits the stress applied to the hair during relaxing treatments such that it does not exceed the fracture stress. Because the only significant stress induced on the hair is due to the axial force applied during thermal relaxing, the hair can once again be modeled as a mechanical spring and relevant equations can be used. According to spring theory, the stresses produced by such a force will always be shear stresses and can be defined by:

$$\tau = \left(\frac{4\frac{d_{curl}}{d_{app}} + 2}{4\frac{d_{curl}}{d_{app}} - 3} \right) \left(\frac{8F_{th} d_{curl}}{\pi d_{app}{}^{3}} \right) \qquad \text{Equation A.1.15}$$

However, in order to adequately compare this to the fracture strength, this shear stress must be converted to an equivalent stress via a von Mises relation. Since the shear is the only stress component present, this formula simplifies to

$$\sigma_{eff} = \sqrt{3}\tau \qquad \text{Equation A.1.16}$$

Now, substituting Equation A.1.15 into A.1.16 and ensuring that it does not exceed the fracture strength, this inequality is defined as:

$$g_2 = \sqrt{3} \left(\frac{4\frac{d_{curl}}{d_{app}} + 2}{4\frac{d_{curl}}{d_{app}} - 3} \right) \left(\frac{8F_{th} d_{curl}}{\pi d_{app}{}^{3}} \right) - \sigma_{frac} \leq 0 \qquad \text{Equation 2.3.17}$$

The remaining inequality constraints are all practical constraints, which set simple upper and lower bounds. All of the following constraints are based on research found in journals, on the Internet, or through interviews with hairstylists that are referenced at the end of this document:

$$g_3 = 0.05 - m_r \le 0$$
$$g_4 = m_r - 0.15 \le 0$$
$$g_5 = 600 - t_r \le 0$$
$$g_6 = t_r - 1200 \le 0$$
$$g_7 = 1/8 - f_r \le 0$$
$$g_8 = f_r - 1/4 \le 0$$
$$g_9 = 390 - T_{th} \le 0$$
$$g_{10} = T_{th} - 475 \le 0$$
$$g_{11} = 0.5 - t_{th} \le 0$$
$$g_{12} = t_{th} - 3 \le 0$$
$$g_{13} = 1/2 - f_{th} \le 0$$
$$g_{14} = f_{th} - 7 \le 0$$
$$g_{15} = 0.098 - F_{th} \le 0$$
$$g_{16} = F_{th} - 0.392 \le 0$$

Design Variables and Parameters

The table below indicates the design variables involved in the objective function and constraints along with typical values for these quantities. All of the values are based on research from journal articles, the Internet, or hairstylist interviews. Assuming that none of the inequality constraints are active and that all equality constraints are active, the model has potentially five degrees of freedom.

Symbols	Meaning	Typical Values
m_r	Quantity of relaxer (mass)	0.05-0.15 kg
t_r	Duration time of relaxer application	600-1200 s
f_r	Frequency of relaxer application	0.125-0.25 applications/week
t_{th}	Duration time of thermal relaxing	0.5-3 s
f_{th}	Frequency of thermal relaxing	0.25-7 applications/week
T_{th}	Temperature applied during thermal relaxing	390-475 K
F_{th}	Force applied during thermal relaxing	0.098-0.392 N
N	Number of curls along effective hair length	5-32 curls
d_{curl}	Curl diameter	0.002-0.012 m
σ_{frac}	Fracture stress of hair	140 MPa

Table A.1.2 Effective Hair Length Subsystem Variables

The next table indicates the design parameters involved in the objective function and constraints along with typical values for these quantities. Again, all of the values are based on research from journal articles, the Internet, or hairstylist interviews.

Symbols	Meaning	Typical Values
$\Delta L_{eff,max}$	Maximum change in effective hair length	0.050 m
$L_{eff,0}$	Initial effective hair length	0.100 m
C_1	Exponential coefficient for curve diameter relation	0.00182 m
C_2	Exponential coefficient for curve diameter relation	0.000182 m
C_3	Exponential coefficient for curl relation	29 curls
C_4	Exponential coefficient for curl relation	3 curls
T_{room}	Ambient temperature during thermal relaxing	300 K
T_{bp}	Boiling point of water	373 K
k	Thermal conductivity of hair	0.040 W/m*K
c_v	Specific heat of water	4184 J/kg*K
L	Actual hair length	0.150 m
d_{app}	Apparent circular diameter of hair	73.6 μm
A	Area of flat iron plate coverage	0.00194 m^2
ρ_{hair}	Area density of hair	187 hairs/cm^2
ρ_{H20}	Mass density of water	1000 kg/m^3
C_5	Chemical relaxer cost sensitivity coefficient	$1*week/kg*application
C_{relax}	Cost of professional chemical relaxing	$50

Table A.1.3 Effective Hair Length Subsystem Parameters

A.1.3 Summary Model

The subsystem design optimization problem can be summarized accordingly in standard, negative null form:

$$\min\ -\Delta L_{eff} = \frac{-\Delta L_{eff,\max}}{2}\left(2 - e^{-m_r t_r f_r} - e^{-T_{th} t_{th} f_{th} F_{th}}\right) + L(1 - e^{-.17 T_{th} t_{th} f_{th} - 1.7 m_r t_r f_r})$$

subject to $h_1 = \pi N d_{curl} - L = 0$

$$h_2 = d_{curv} - C_1 e^{m_r t_r f_r} - C_2 e^{T_{th} t_{th} f_{th} F_{th}} = 0$$

$$h_3 = N - C_3 e^{-m_r t_r f_r} - C_4 e^{-T_{th} t_{th} f_{th} F_{th}} = 0$$

$$h_4 = \sigma_{frac} - 57 d_{curl} - 129 = 0$$

$$h_5 = \frac{2\pi L k\left(T_{th} - T_{room}\right)}{\rho_{hair} A \ln\left(\frac{0.5 d_{app}}{0.5 d_{app} - 0.5x10^{-6}}\right)} - \frac{0.25 \rho_{H20} \pi \left(d_{app} - 1x10^{-6}\right)^2 L c_v (T_{bp} - T_{room})}{t_{th}} = 0$$

$$g_1 = C_5 m_r f_r - C_{relax} \le 0$$

$$g_2 = \sqrt{3}\left(\frac{4\frac{d_{curl}}{d_{app}} + 2}{4\frac{d_{curl}}{d_{app}} - 3}\right)\left(\frac{8 F_{th} d_{curl}}{\pi d_{app}^{\ 3}}\right) - \sigma_{frac} \le 0$$

$$g_3 = 0.05 - m_r \le 0$$

$$g_4 = m_r - 0.15 \le 0$$

$$g_5 = 600 - t_r \le 0$$

$$g_6 = t_r - 1200 \le 0$$

$$g_7 = 1/8 - f_r \le 0$$

$$g_8 = f_r - 1/4 \le 0$$

$$g_9 = 390 - T_{th} \le 0$$

$$g_{10} = T_{th} - 475 \le 0$$

$$g_{11} = 0.5 - t_{th} \le 0$$

$$g_{12} = t_{th} - 3 \le 0$$

$$g_{13} = 1/2 - f_{th} \le 0$$

$$g_{14} = f_{th} - 7 \le 0$$

$$g_{15} = 0.098 - F_{th} \le 0$$

$$g_{16} = F_{th} - 0.392 \le 0$$

A.1.4 Model/Monotonicity Analysis

Due to time constraints and multiple reconfigurations of the above model for such a complicated optimization problem, no detailed monotonicity analysis has been completed to date. Many issues regarding infeasibility of the model have limited a thorough monotonicity analysis and indicate that further reconfiguration of the model may be necessary. However, an initial start to such a process has been documented in the monotonicity table below. Note that for equality constraints the signs in the parentheses indicate that the constraint has been directed such that h≥0.

	m_r	t_r	f_r	T_{th}	t_{th}	f_{th}	F_{th}	N	d_{curl}	σ_{frac}
f	U	U	U	U	U	U	-			
h_1								+ (-)	+ (-)	
h_2	- (+)	- (+)	- (+)	- (+)	- (+)	- (+)	- (+)		+ (-)	
h_3	+ (-)	+ (-)	+ (-)	+ (-)	+ (-)	+ (-)	+ (-)	+ (-)		
h_4									- (+)	+ (-)
h_5				+ (-)	+ (-)					
g_1	+		+							
g_2							+		U	-
g_3	-									
g_4	+									
g_5		-								
g_6		+								
g_7			-							
g_8			+							
g_9				-						
g_{10}				+						
g_{11}					-					
g_{12}					+					
g_{13}						-				
g_{14}						+				
g_{15}							-			
g_{16}							+			

Table A.1.4 Monotonicity Table for Change in Effective Hair Length

A.1.5 Numerical Algorithms/Results

Due to time constraints, multiple reconfigurations of the above model, and a failed monotonicity analysis for such a complicated optimization problem, no numerical algorithm has been employed thus far to solve this problem. Many issues regarding infeasibility of the model have limited a thorough monotonicity analysis, which in turn has hindered progress in terms of numerical solutions. This indicates that further reconfiguration of the model may be necessary. Potential items that could go into a parametric study would include any of the exponential coefficients in the model as well as the cost sensitivity coefficient.

A.2 Previous Hair Styling Subsystem work

A.2.1 Nomenclature

Symbols	Meaning	Units
V_{dye}	Volume of dye	L
t_{dye}	Duration time of dye application	min
f_{dye}	Time between dye application	days
A_{braid}	Cross-sectional area of braid	cm^2
W_{braid}	Load on hair due to braid extensions	N
Q_{braid}	Qty of braids	-
t_{braid}	The length of time client keeps the braid style in	weeks
f_{braid}	How often client gets hair braided	#times/year
l_{braid}	Length of braids	m
$t_{th,styling}$	Average time one applies heat to hair during a styling session	min
$f_{th,styling}$	Frequency of thermal styling; how often one uses these appliances	#times/day
$T_{th,styling}$	Temperature setting during thermal styling	K
$q_{th,styling}$	Heat (energy) during thermal styling	W
l_{hair}	Length of hair client starts process with	cm
l_{ng}	Length of new-growth (un-dyed hairs that has grown in since last visit)	cm
A_{scalp}	Surface area of scalp	cm^2
ρ_{hair}	Area density of hair	hairs/cm^2
dl/dt	Hair growth rate	μm/day
E	Young's Modulus	MPa
m_{frac}	Mass to break healthy hair	g
σ_{frac}	Fracture strength	Mpa
σ_{braid}	Stress due to braids	Mpa
ε_{frac}	Fracture strain	%
ε_{braid}	Strain due to braids	%
A_{hair}	Cross-sectional area of client hair	μm^2
g	Gravity	m/s^2
K	Hair density and diameter quality coefficient	-
Z	Braid size and time it lasts relation	-
C_{dye}	Cost for getting dye treatment	$
$C_{styling}$	Cost for getting thermal styles	$
C_{braids}	Cost for getting braids	$

Table A.2.1 Hair Styling Subsystem Nomenclature

A.2.2 Mathematic Models

Changes since First Draft

The styling sub-function demanded a qualitative understanding of the relationships between the parameters and the variables. A key relationship is that between the cross-sectional area of strand of hair (μm^2) and the density of the hair (#hairs/cm^2). Table 3.3.1 characterizes the relative relationship between these two parameters. Each of the values within the chart represents a hair quality coefficient that will be used to develop the model. The values for general hair diameters and values for hair density, were taken from Zviak and Bouillon, 1986 and Loussuran, et al, 2005, respectively.

The relationship for K was computed simply by multiplying D_{hair} by ρ_{hair}. A constant was include to remove the dimensions.

	Density (#hairs/cm^2)		
Hair diameter (μm)	**Low** (less than 200)	**Avg.** (200 – 300)	**High** (greater than 300)
Fine (less than 50 μm)	9751	12,250	14,749
Avg (60 – 80 μm)	13,930	17,500	21,070
Thick (over 80 μm)	16,119	20,250	24,381

Table 3.3.1 Values for hair quality coefficient, K

Based on the values obtained, the following range of values can be interpreted to have the following meaning:

Less susceptible to effects of styling: $17{,}500 \leq K \leq 24{,}381$

Borderline: $14{,}749 \leq K \leq 17{,}500$

More susceptible to effects of styling: $9{,}751 \leq K \leq 14{,}749$

<u>Changes since Progress Report</u>

Objective Function

$$\min f(\mathbf{x};\mathbf{p}) = |\mathbf{y}|^2$$
where
$$\mathbf{x} = (V_{dye}, t_{dye}, f_{dye}, A_{braid}, W_{braid}, Q_{braid}, t_{braid}, f_{braid}, l_{braid})^T$$
$$\mathbf{p} = (l_{hair}, l_{ng}, A_{scalp}, \rho_{hair}, dl/dt, \sigma_{frac}, \sigma_{braid}, \varepsilon_{frac}, \varepsilon_{braid}, A_{hair}, m_{frac}, E, K, Z)^T$$

where y represents the styles recommended for the client based on nominal values and values that represent the desires of the client (in the numerator). The function will calculate how much the "desired" and "recommended" styles deviate from one another and minimizing this deviation means meeting the desires of the client as best as possible while keeping in mind the health and progress of the hair. For simplicity, this function will be projected out to one year from the time of consultation with the client with the understanding that the recommendations can change as client is diligent to keep the system updated. The expression for y is as follows:

$$y = \left(\frac{s_{dye}}{s_{dye,nom}}\right) + \left(\frac{s_{braid}}{s_{braid,nom}}\right) + \left(\frac{s_{th}}{s_{th,nom}}\right) \qquad \text{Equation A.2.1}$$

where:

$$s_{dye,nom} = Kt_{dye}e^{-A_{hair}\rho_{hair}f_{dye}t_{dye}} + e^{-V_{dye}} \qquad \text{Equation A.2.2}$$

$$s_{braid,nom} = W_{braid}Q_{braid} \qquad \text{Equation A.2.3}$$

The styling sub-function demanded a qualitative understanding of the relationships between the parameters and the variables. A key relationship is that between the cross-sectional area of strand of hair (μm^2) and the density of the hair (#hairs/cm^2). Table 3.3.1 characterizes the relative relationship between these two parameters. Each of the values within the chart represents a coefficient that will be used to develop the model. The values for general hair diameters and and values for hair density, were taken from Zviak and Bouillon, 1986 and Loussuran, et al, 2005, respectively.

	Density (#hairs/cm^2)		
Hair diameter (µm) Rating:	**Low** (less than 200) 1	**Avg.** (200 – 300) 0.5	**High** (greater than 300) 0.25
Fine (less than 50 µm) Rating: 2	2	1	0.5
Avg (60 – 80 µm) Rating: 1	1	0.5	0.25
Thick (over 80 µm) Rating: 0.5	0.5	0.25	0.125

Table A.2.1 Definition of Coefficient 'K' based on Hair Diameter and Density

In addition, the qualitative relationship between size of braids and time in which one can keep them in is best characterized as shown in Table 3.3.2. This assumes someone with a healthy head of hair. The scale used goes from -2 to +2, where -2 means "not feasible/atypical" and +2 means "feasible/typical" which is representative of observations in many women. In addition, this may vary per choice of the person wearing it where a person with micro braids may in fact decide to redo their hair every 2 weeks and not mind sitting for another 12 hours to do them all over again. In addition, another female may be content with wearing large braids for longer than 1 month without redoing it, even if there is a lot of fly-away hair strands which thus takes away from the beauty of the style.

	Time			
Size of braid (cm^2)	**Often (bi-weekly)**	**Moderate (monthly)**	**Occasionally (every 6 months)**	**Hardly Ever (yearly)**
"Micro"braids (less than 10)	-2	-1	+2	+1
Medium braids $10 \leq A_{braid} \leq 50$	+1	+2	-1	-2
Large braids $A_{braid} \geq 50$	+2	+1	-1	-2

Table A.2.2 Relationship between braid size and time in which it can be kept in, represented as 'Z'

Based on this relationship, it can be stated that the smaller the size of the braids, the longer a person can keep the style in (ie. The longer the style lasts). This is due to the fact that the surface area of small braids is less than those of large braids. When the surface area is small, that means there are more opportunities for ones own hair to be neatly tucked in to the brain pattern. Larger braids mean that there is more opportunity for hair to be pulled out of the braid thus decreasing the time that it can stay in. As a result, the following relationships are true:

$$A_{braid} \propto 1/t_{braid} \qquad \text{Equation A.2.4}$$

$$t_{braid} \propto 1/f_{braid} \qquad \text{Equation A.2.5}$$

As the size of the braid increases, the time that the style can be kept decreases. As the time in which you can keep a braid style decreases, this means you will have to get the style done again more frequently (ie. f_{braid} increases).

In developing this model, there are some useful formulations that can be used in the analysis.

The new growth function represents the amount of hair that grows on a daily basis. Multiplying the frequency in which dye should be applied (ensuring that time units are the same) gives the new growth function.

New-growth function:

$$l_{ng} = \left(\frac{dl}{dt}\right) * f_{dye} \qquad \text{Equation A.2.6}$$

The total volume of dye necessary for one year of dye styling consists of calculating the total surface area of hair on the client's scalp during the first visit. Then, for each subsequent visit (most likely every 2 months or more), dye is applied to the new growth to keep the hair in an even or well blended color. This is represented by the two expressions below for volume.

Volume of dye (1st visit):

$$V_{dye,1/cm^2} = 0.001 A_{scalp} (A l_{hair} \rho_{hair}) \qquad \text{Equation A.2.7}$$

Volume of dye (on new-growth): $V_{dye,ng/cm^2} = 0.001 A_{scalp}(A l_{ng} \rho_{hair})$ Equation A.2.8

The weight of braided hair must be calculated on a per unit length basis since the length of braids can be varied. It was assumed that braids are sold in units of ounces which were the converted to grams.

Weight of braid (per unit/length): $W_{braid} = \frac{28.3}{meter} l_{braid}$ → (1 oz = 28.3 grams) Equation A.2.9

This formulation ensures that the recommended braid style output (includes recommended cross-sectional area of the braids), provides a quantity that is equivalent to available surface area of the scalp divided by the area of the braids themselves.

Quantity of braids defined: $Q_{braids} = \frac{A_{scalp}}{A_{braids}}$ Equation A.2.10

In addition, over a long period of time, there is a degree of strain that the client's hair may undergo due to length of the braid and load applied to the hair from the extension. Strain from the braids must not exceed fracture strain. The formula for the strain due to the braids is given by:

Strain from braids: $\varepsilon_{braid} = \frac{W_{braid}}{A_{braid} E}$ Equation A.2.11

Braids add stress to the hair. Like strain, stress due to braids must not exceed fracture stress. The formulation below is the stress that braiding itself puts on the hair.

Stress from braids*:* $\sigma_{braid} = \frac{W_{braid}}{A_{braid}}$ Equation A.2.12

Based on the Table of constants defined above, it was necessary to formulate a relationship between the constants, size of braid and time.

***Frequency of braids relation*:** $$f_{braid,rel} = KZ\frac{A_{braid}}{t_{braid}}$$ Equation A.2.13

A formulation the then express the frequency of braids using the above relation is desirable since it is important to consider frequency in terms of the value achieved for the braid size (A_{braid}) design variable. For the size will determine the frequency. Therefore,

$$f_{braid} = f(f_{braid,rel})$$ Equation A.2.14

Additional formulations will be developed using other relevant relationships, a thorough review of the literature and other sources and perhaps interviews with scientists at beauty supply manufacturers will be instrumental in this process.

Constraints

There are several constraints necessary to properly define the problem. The first one consists of ensuring that the strain imposed by the braids does not exceed the fracture strain that is characteristic of the client's hair. The constraint g_1 is given by:

$$g_1 : \varepsilon_{braid} \leq 0.8\varepsilon_{frac}$$ Equation A.2.15

The constraint g_2 below provides a limitation that protects the client from getting a style that exceeds what their own unique hair can actually handle.

$$g_2 : \sigma_{braid} \leq 0.8\sigma_{frac}$$ Equation A.2.16

It is stated that hair can fracture at a mass of 50 – 100 grams for healthy hair. Therefore, a limit is placed on the weight of the braids to not exceed 50% of the force applied due to force that can cause fracture. (Note: g = gravity)

$$g_3 : W_{braid} \leq 0.5 m_{frac} g \qquad \text{Equation A.2.17}$$

In addition, a constraint is placed on the frequency of getting braids relationship. The constraint below represents those boundaries that protects worst case scenario clients and braid conditions.

$$g_4 : 20 \leq f_{braid,rel} \leq 100 \qquad \text{Equation A.2.18}$$

Additional constraints will have to be developed that take into account the cost associated with styling techniques highlighted above. There are some additional relationships that need to be developed to represent this model in the most practical way as possible. In addition, appropriate simplifying assumptions will have to be employed as well.

Design Variables and Parameters

In order to make the best recommendations to the client, it will be necessary to assess the feasibility of implementing certain styles and processes on the client's hair, while keeping hair health in mind. The key design variables and parameters will include the following:

Symbols	Meaning	Typical Values
V_{dye}	Volume of dye	Depends on hair
t_{dye}	Duration time of dye application	20-30 min
f_{dye}	Time between dye application	Every two months
A_{braid}	Cross-sectional area of braid	Varies
W_{braid}	Load on hair due to braid extensions	N/A
Q_{braid}	Qty of braids	Varies from 1-500
l_{braid}	Length of braids	0.25 – 1 m
t_{braid}	The length of time client keeps the braid style in	1 wk-6 months
f_{braid}	How often client gets hair braided	Varies
$t_{th,styling}$	Average time one applies heat to hair during a styling session	min
$f_{th,styling}$	Frequency of thermal styling; how often one uses these appliances	daily
$T_{th,styling}$	Temperature setting during thermal styling	422-478 K
$q_{th,styling}$	Heat (energy) during thermal styling	N/A

Table A.2.3 Hair Styling Subsystem Variables

Symbols	Meaning	Typical Values
l_{hair}	Length of hair client starts process with	Varies
l_{ng}	Length of new-growth (un-dyed hairs that has grown in since last visit)	Varies
A_{scalp}	Surface area of scalp	cm^2
ρ_{hair}	Density of hair	187 hairs/cm^2
dl/dt	Hair growth rate	μm/day
E	Young's Modulus	1243 MPa
m_{frac}	mass to break healthy hair	50 – 100 g
σ_{frac}	Fracture stress	140 MPa
σ_{braid}	Stress due to braids	N/A
ε_{frac}	Fracture strain	45 %
ε_{braid}	Strain due to braids	N/A
A_{hair}	Cross-sectional area of client hair (ellipse formulation)	16930 μm^2
g	gravity	m/s^2
K	Hair density and diameter quality coefficient	-
Z	Braid size and time it lasts relation	-
C_{dye}	Cost for getting dye treatment	$70
$C_{styling}$	Cost for getting thermal styles	$30
C_{braids}	Cost for getting braids	$50-300

Table A.2.4 Hair Styling Subsystem Parameters

A.2.3 Summary Model

In closing, the model can be summarized as follows:

$$\min f(\mathbf{x};\mathbf{p}) = |\mathbf{y}|^2$$

where

$$\mathbf{x} = (V_{dye}, t_{dye}, f_{dye}, A_{braid}, W_{braid}, Q_{braid}, t_{braid}, f_{braid}, l_{braid})^T$$

$$\mathbf{p} = (l_{hair}, l_{ng}, A_{scalp,} \rho_{hair}, dl/dt, \sigma_{frac}, \sigma_{braid}, \varepsilon_{frac}, \varepsilon_{braid}, A_{hair}, m_{frac}, E, K, Z)^T$$

subject to:

$$g_1 : \varepsilon_{braid} < \varepsilon_{frac}$$

$$g_2 : \sigma_{braid} < \sigma_{frac}$$

$$g_3 : W_{braid} \leq 0.5 m_{frac} g$$

$$g_4 : 20 \leq f_{braid,rel} \leq 100$$

B. Relevant MatLab Code

B.1 Hair Length Code

```
%Code Developed by:  Michael Alexander
%Project: African-American Hair Beauty Study
%Filename:  malexanzFUN3.m
%Purpose of File: Objective Function for Effective Length Subsystem
%April 2, 2007

function[f]=malexanzFUN3(x)

%x(1)=mr  (kg)
%x(2)=tr  (s)
%x(3)=fr  (1/s)
%x(4)=Tth (K)
%x(5)=tth (s)
%x(6)=fth (1/s)

%Function Parameters

DLeffmax=0.05;      %Maximum change in effective length (meters)
C1=10000;           %Exponential argument scaling coefficient
C2=2;               %Chemical relaxing scaling coefficient

%Unscaling variables for constraint computation (currently not used)
%lb=[0.05 600 0.000000207 390 1 0.000000827];

%ub=[0.15 1200 0.000000413 475 3 0.000011574];

%D=eye(6);
%s=zeros(6,1);

%for j=1:1:6
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%x=(inv(D)*(x'-s))';

f=-(DLeffmax/2)*(2-exp(-C1*C2*x(1)*x(2)*x(3))-exp(-C1*x(4)*x(5)*x(6)));
```

```matlab
%Code Developed by:  Michael Alexander
%Project: African-American Hair Beauty Study
%Filename:  malexanzNONLCON3.m
%Purpose of File: Nonlinear Constraints for Effective Length Subsystem
%April 2, 2007


function[g,h]=malexanzNONLCON3(x)

%Function parameters

C2=2;                %Chemical relaxing scaling coefficient
C3=1;                %Chemical relaxer cost sensitivity coefficient
C4=1.5;              %Chemical/thermal relaxing time sensitivity coefficient
C_relax=50;          %Cost of professional chemical relaxing
T_room=300;          %Ambient temperature during thermal relaxing (K)
T_bp=373;            %Boiling point of water
k=0.04;              %Thermal conductivity of hair (W/m*K)
c_v=4184;            %Specific heat of water (J/kg*K)
L=0.15;              %Actual hair length
d_app=73.6e-6;       %Apparent circular diameter of hair (m2)
A=0.00194;           %Area of flat iron plate coverage (m2)
rho_hair=187;        %Area density of hair (hairs/cm2)
rho_H20=1000;        %Mass density of water (kg/m3)

%Unscaling variables for constraint computation (currently not used)

%lb=[0.05 600 0.000000207 390 1 0.000000827];

%ub=[0.15 1200 0.000000413 475 3 0.000011574];

%D=eye(6);
%s=zeros(6,1);

%for j=1:1:6
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%x=(inv(D)*(x'-s))';

%Nonlinear inequality constraints
%g(1)=g1
%g(2)=g2
%g(3)=g3
%g(4)=g4
g=[(C3*x(1)*x(3))-C_relax;(C4*x(2)*x(3))-0.0002484;(C4*x(5)*x(6))-
0.000011574;(C2*x(1)*x(4))-58.5];

%Nonlinear equality constraints
%h=h1
h=[((2*pi*L*k*(x(4)-T_room))/(rho_hair*1e4*A*log(0.5*d_app/(0.5*d_app-(0.5e-
6)))))-...
    ((0.25*rho_H20*pi*((d_app-(1e-6))^2)*L*c_v*(T_bp-T_room))/x(5))];
```

```
%Code Developed by:  Michael Alexander
%Project: African-American Hair Beauty Study
%Filename:  Leff3.m
%Purpose of File: Execution of Optimization Algorithm for Effective Length
Subsystem
%April 2, 2007

clear all

format long e;

%Setting output and large scale to off
options=optimset('Display','iter','LargeScale','off','ActiveConstrTol',1e-
10,'TolCon',1e-10);

%Matrix/vectors for linear constraints (not used)
A=[]; b=[]; Aeq=[]; beq=[];

%Lower bounds
%lb(1)=g6
%lb(2)=g8
%lb(3)=g10, converted to inverse seconds
%lb(4)=g12
%lb(5)=g14
%lb(6)=g16, converted to inverse seconds
lb=[0.05 600 0.000000207 390 1 0.000000827];

%Upper bounds
%ub(1)=g7
%ub(2)=g9
%ub(3)=g11, converted to inverse seconds
%ub(4)=g13
%ub(5)=g15
%ub(6)=g17, converted to inverse seconds
ub=[0.15 1200 0.000000413 475 3 0.000011574];

%Starting Point
x0=ones(1,6);

%Scaling parameters for variables (currently not used)
%D=eye(6);
%s=zeros(6,1);

%for j=1:1:6
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%y0=((D*x0')+s)';

%Optimization Algorithm
[xopt,fval,exitflag,output,lambda]=fmincon('malexanzFUN3',x0,A,b,Aeq,beq,lb,ub,
'malexanzNONLCON3',options)
```

B.2 Hair Styling Code

```
%Code Developed by:  Tahira Reid
%Project: African-American Hair Beauty
%Subsystem:  Styling
%Filename:  styleFUN.m
%Purpose:  Objective function file

function [y]=styleFUN(x)

C=01;

%objective function

y=C*(1-exp(-x(1)*x(2)*x(3))+1-exp(-x(4)*x(5)*x(6))+1-exp(-
x(9)*x(7)+0.8*x(8)*x(9)));

%x(1)=tdye, processing time for dye
%x(2)=fdye, frequency at which client can get their hair dyed
%x(3)=Vdye, volume of dye to be used on the client's hair
%x(4)=lbraid, total length of braid which includes extensions
%x(5)=Dbraid, diameter of the braid
%x(6)=fbraid, frequency at which braid style is redone
%x(7)=ftsp, frequency at which client has thermal styles done at salon
%x(8)=ftsc, frequency at which client has thermal styles done on her own
%x(9)=Tts, temperature at which thermal styles are done


%Code Developed by:  Tahira Reid
%Project: African-American Hair Beauty
%Subsystem:  Styling
%Filename:  styleNONLCON.m
%Purpose:  Constraints file

function [g,h]=styleNONLCON(x)

%Parameters of our imaginary client.  She comes in and has the following
%hypothetical values based on literature.

%Diameter of a single hair strand (micrometers)
Dhair = 73.4;

%Area density of her hair (number of hairs per centimeters squared)
rhohair=187;

%Hypothetical length of hair she starts the process with (inches)

lhair=3*2.54;   %hair length converted from inches to microns


%Conversion factor to make K dimensionless
sk=1;%/100;%(1/10^4);
```

```
%Client's hair quality factor computed
K=rhohair*Dhair*sk;

%Convert minutes into weeks
a=1/(60*24*7);

 %Acceptable range of values for (lower and upper bounds);
 Kd=[9751 24381]*sk; %hair quality constant based on client parameters

tdye=[15 30]*a;
fdye=[1/52 1/4];
Vdye=[29.6 118.3]*01;

lbraid=[2*lhair 8*lhair]*01;
Dbraid=[0.3175 1.5785];
fbraid=[1/52 1/4];

ftsp=[1/8 1];
ftsc=[1/2 7];
Tts=[386 483];%*0.01;

%Constraints that relate the diameter of the braid (x5) and length of the
%braid (x4)
g1=-x(4)*x(5)+lbraid(2)*Dbraid(1);
g2=x(4)*x(5)-lbraid(1)*Dbraid(2);

%Constraints that relate the diameter of the braid (x5) with the frequency
%at which the hair should be braided (x6)
g3=-x(5)*x(6)+Dbraid(1)*fbraid(1);
g4=x(5)*x(6)-Dbraid(2)*fbraid(2);

%Constraints that relate the frequency of thermal styling done at a salon
%(x7) and at home (x8) with x(9), the typical temperatures
g5=-x(9)*x(7)+Tts(2)*0.799*1;
g6=x(9)*x(7)-Tts(2)*0.799*7;

g7=-x(9)*x(8)+Tts(2)*(1/8);
g8=x(9)*x(8)-Tts(2)*(1/2);

% *x(8)+Tts(1)*ftsp(1)*ftsc(1);
% g6=x(9)*x(7)*x(8)-Tts(2)*ftsp(2)*ftsc(2);

g=[g1;g2;g3;g4;g5;g6;g7;g8];
h=[];
```

```
%Code Developed by:  Tahira Reid
%Project: African-American Hair Beauty
%Subsystem:  Styling
%Filename:  stylemainfile.m
%Purpose:  Execution file

clc
clear variables
format long e


x0=[1,1,1,1,1,1,1,1,1];

A=[];
b=[];

Aeq=[];
beq=[];

%Factors used to test the effects of scaling target variables

sdye=1;
sbraid=1;
sTts=1;  %0.01

lb=[15, 1/52, 29.6*sdye, 15.24*sbraid, 0.3175, 1/52, 1/8, 1/2, 386*sTts];
ub=[30, 1/4, 118.3*sdye, 60.96*sbraid, 1.5785,  1/4,  1, 7, 483*sTts];



[x,fval,exitflag,output,lambda,gradient]=fmincon('styleFUN',x0,A,b,Aeq,beq,lb,u
b,'styleNONLCON')
```

B.2.1 Results

```
 arning: Large-scale (trust region) method does not currently solve this type
of problem,
 using medium-scale (line search) instead.
> In fmincon at 303
  In stylemainfile at 32
Optimization terminated: first-order optimality measure less
 than options.TolFun and maximum constraint violation is less
 than options.TolCon.
Active inequalities (to within options.TolCon = 1e-006):
  lower      upper     ineqlin   ineqnonlin
    6          2                      1
               7

x =

  Columns 1 through 3

    1.500010000000000e+001    2.500000000000000e-001    2.960010000000000e+001
```

```
  Columns 4 through 6

    1.524110065540124e+001    1.269908285340332e+000    1.923076923076923e-002

  Columns 7 through 9

    1.000000000000000e+000    5.192014120999901e-001    3.861002968408988e+002


fval =

    2.310788916182229e+000


exitflag =

     1


output =

       iterations: 6
        funcCount: 70
     lssteplength: 1
         stepsize: 1.952473962913740e-010
        algorithm: 'medium-scale: SQP, Quasi-Newton, line-search'
    firstorderopt: 2.333348542160607e-008
          message: [1x144 char]


lambda =

         lower: [9x1 double]
         upper: [9x1 double]
         eqlin: [0x1 double]
      eqnonlin: [0x1 double]
        ineqlin: [0x1 double]
    ineqnonlin: [8x1 double]


gradient =

                         0
                         0
                         0
    1.683143920067945e-002
    2.020064456576660e-001
    1.333954077959061e+001
                         0
                         0
                         0
```

B.3 Hair Damage Code

```
%Code Developed by:  Harshit Sarin
%Project: African-American Hair Beauty Study
%Subsystem Name:  Hair Damage
%Filename:  stressFUN.m
%Purpose of File: Objective function definition

function [f] = stressFUN(x)

% Parameter definitions
Cth = 3;
Cbr = 3;
Ahair = 0.016930;
Cco = 3;
Kch = 100;
deltat_ch = 1.6;
tau_ch = 14;
Cch = 3;

%Objective function
f = 0.17*x(1)*(1+exp(-Cth/x(2)))-(0.43*x(3)-0.013*x(3)^2)*(1+exp(-
Cbr/x(4)))+(x(5)/Ahair)*(exp(-Cco/x(6)))+ Kch*(1-exp(-(x(7)-
deltat_ch)/tau_ch))*(1+exp(-Cch/x(8)));


%Code Developed by:  Harshit Sarin
%Project: African-American Hair Beauty Study
%Subsystem Name:  Hair Damage
%Filename:  stressNONLCON.m
%Purpose of File: Constraint definition

function [g,h] = stressNONLCON(x)

Ahair = 0.016930;

% Constraints
g1 = 393 - x(1);
g2 = 0.33-x(2);
g3 = 0.5-x(9);
g4 = -5+x(9);
g5 = 0.33-x(4);
g6 = -30+x(4);
g7 = -2*x(4)+x(2);
g8 = 2-x(6);
g9 = 0.1-x(5);
g10 = 0.33-x(8);
g11 = -2*x(4)+x(8);
g12 = 15-x(7);
g13 = 518.9-247.7*x(5)+119.5*x(5)^2-x(1);
g14 = 34.65-53.5*x(5)+36.46*x(5)^2-x(7);

g = [g1;g2;g3;g4;g5;g6;g7;g8;g9;g10;g11;g12;g13;g14];
```

```
h = [x(3)-1380*Ahair*x(9)];


%Code Developed by:  Harshit Sarin
%Project: African-American Hair Beauty Study
%Subsystem Name:  Hair Damage
%Filename:  stress_prob.m
%Purpose of File: Main file for execution

clear all
clc
format long e;

% Variable Matrix

%x(1) = Temp-thermal
%x(2) = freq-thermal
%x(3) = twist factor braid
%x(4) = freq-braid
%x(5) = Force-comb
%x(6) = freq-comb
%x(7) = time-chemical
%x(8) = freq-chemical
%x(9) = A-braid

A=[];
B=[];
Aeq=[];
Beq=[];
lb=[0,0,0,0,0,0,0,0,0]; %bounds are in the non-linear constraint set
ub=[];
[x,fval,exitflag,output,lambda,grad,hessian] =
fmincon('stressFUN',[100,100,100,100,100,100,100,100,100],A,B,Aeq,Beq,lb,ub,'st
ressNONLCON')
```

B.3.1 Results

```
Active inequalities (to within options.TolCon = 1e-006):
  lower      upper     ineqlin   ineqnonlin
                                     2
                                     6
                                     8
                                    10
                                    13
                                    14
```

xopt =

Columns 1 through 3

```
   4.017587574726774e+002    3.300000000000000e-001    1.653845252772258e+001

  Columns 4 through 6

   3.000000000000000e+001    7.300244789962314e-001    2.000000000000000e+000

  Columns 7 through 9

   1.502452745105965e+001    3.300000000000000e-001    7.078786703871332e-001
```

fval =   1.328302537616078e+002

exitflag =   1

lambda =

```
         lower: [9x1 double]
         upper: [9x1 double]
         eqlin: [0x1 double]
      eqnonlin: 8.435837214980743e-010
       ineqlin: [0x1 double]
    ineqnonlin: [14x1 double]
```

lambda1 = 0
lambda2 = 0.212018966674805
lambda3 = 0
lambda4 = 0
lambda5 = 0
lambda6 = 0.010724639892578
lambda7 = 0
lambda8 = 7.216057777404785
lambda9 = 0
lambda10 = 0.191436767578125
lambda11 = 0
lambda12 = 0
lambda13 = 0.170019135539699
lambda14 = 2.738281915575873

B.4 Hair Beauty Code

```
%Objective Function for Hair Beauty System
%Michael Alexander, Tahira Reid, Harshit Sarin
%ME 555
```

```
%April 14, 2007

function[f]=beautyFUN(x)

%x(1)=mr  (kg)
%x(2)=tr  (s)
%x(3)=fr  (1/s)
%x(4)=Tth (K)
%x(5)=tth (s)
%x(6)=fth (1/s)
%x(7)=tau ()
%x(8)=fbraid (#times/month)
%x(9)=Fcomb (N)
%x(10)=fcomb (#times/day)
%x(11)=Abraid (#turns/cm)
%x(12)=tdye, processing time for dye
%x(13)=fdye, frequency at which client can get their hair dyed
%x(14)=Vdye, volume of dye to be used on the client's hair
%x(15)=lbraid, total length of braid which includes extensions
%x(16)=Dbraid, diameter of the braid

%Function Parameters for Effective Hair Length

DLeffmax=0.05;      %Maximum change in effective length (meters)
C1=10000;           %Exponential argument scaling coefficient
C2=2;               %Chemical relaxing scaling coefficient

%Function Parameters for Hair Damage

Cth=3;
Cbr=3;
Ahair=0.016930;
Cco=3;
Kch=100;
deltat_ch=1.6;
tau_ch=14;
Cch=3;
T=30*24*3600;       %Conversion factor for fr,fth, in hair damage function
T2=60;              %Conversion factor for tr in hair damage function
sigma0=254;         %Nominal hair strength

%Function Parameters for Hair Styling

P1=0.01/4;
P2 =0.2/2;
P3 =1;
T3=100;     %(Scaling) conversion factor
T4=4;

%Unscaling variables for constraint computation (currently not used)
%lb=[0.05 900 0.000000207 393 1 0.000000827 11.6871 0.33 0.1 2 0.5 15 0.09123
29.6 15.24 0.3175];

%ub=[0.15 1200 0.000000413 475 3 0.000011574 116.817 4 3 10 5 30 0.25 118.30
60.96 1.5785];
```

```
%D=eye(16);
%s=zeros(16,1);

%for j=1:1:16
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%x=(inv(D)*(x'-s))';

%Effective Hair Length Subfunction
f1=-(DLeffmax/2)*(2-exp(-C1*C2*x(1)*x(2)*x(3))-exp(-C1*x(4)*x(5)*x(6)));

%Hair Damage Subfunction
f2=0.17*x(4)*(1+exp(-Cth/(x(6)*T)))-(0.43*x(7)-0.013*x(7)^2)*(1+exp(-
Cbr/x(8)))+...
    (x(9)/Ahair)*(exp(-Cco/x(10)))+ Kch*(1-exp(-((x(2)/T2)-
deltat_ch)/tau_ch))*(1+exp(-Cch/(x(3)*T)));

%Hair Styling Subfunction
f3=-((1-exp(-x(12)*x(13)*x(14)*P1))+(1-exp(-P2*x(15)*x(16)*(x(8)/T4)))+(1-exp(-
P3*(x(4)/T3)*x(6))));

%All-in-One Objective Function
f=(f1/DLeffmax)+(f2/sigma0)+(f3/3);
*******************************************************************************

%Nonlinear Constraints for Hair Beauty System
%ME 555
%April 14, 2007

function[g,h]=beautyNONLCON(x)

%Function Parameters (Effective Hair Length)

C2=2;               %Chemical relaxing scaling coefficient
C3=1;               %Chemical relaxer cost sensitivity coefficient, nom 1
C4=1.25;             %Chemical/thermal relaxing time sensitivity coefficient,
nom 1.5
C_relax=50;         %Cost of professional chemical relaxing
T_room=300;         %Ambient temperature during thermal relaxing (K)
T_bp=373;           %Boiling point of water
k=0.04;             %Thermal conductivity of hair (W/m*K)
c_v=4184;           %Specific heat of water (J/kg*K)
L=0.15;             %Actual hair length
d_app=73.6e-6;      %Apparent circular diameter of hair (m2)
A=0.00194;          %Area of flat iron plate coverage (m2)
rho_hair=187;       %Area density of hair (hairs/cm2)
rho_H20=1000;       %Mass density of water (kg/m3)

%Unscaling variables for constraint computation (currently not used)
%lb=[0.05 900 0.000000207 393 1 0.000000827 11.6871 0.33 0.1 2 0.5 15 0.09123
29.6 15.24 0.3175];
```

```
%ub=[0.15 1200 0.000000413 475 3 0.000011574 116.817 4 3 10 5 30 0.25 118.30
60.96 1.5785];

%D=eye(16);
%s=zeros(16,1);

%for j=1:1:16
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%x=(inv(D)*(x'-s))';

%Nonlinear Inequality Constraints (Effective Hair Length)
g1=(C3*x(1)*x(3))-C_relax;
g2=(C4*x(2)*x(3))-0.0002484;
g3=(C4*x(5)*x(6))-0.000011574;
g4=(C2*x(1)*x(4))-58.5;

%Nonlinear Equality Constraints (Effective Hair Length)
h1=((2*pi*L*k*(x(4)-T_room))/(rho_hair*1e4*A*log(0.5*d_app/(0.5*d_app-(0.5e-
6)))))-...
    ((0.25*rho_H20*pi*((d_app-(1e-6))^2)*L*c_v*(T_bp-T_room))/x(5));

%Function Parameters (Hair Damage)
Ahair = 0.016930;

%Nonlinear Inequality Constraints (Hair Damage)
g5 = -2*x(8)+x(6);
g6 = -2*x(8)+x(3);
g7 = 518.9-247.7*x(9)+119.5*x(9)^2-x(4);
g8 = 34.65-53.5*x(9)+36.46*x(9)^2-x(2);

%Nonlinear Equality Constraints (Hair Damage)
h2=x(7)-1380*Ahair*x(11);

%Function Parameters (Hair Styling)
Dhair = 73.4;            %Diameter of a single hair strand (micrometers)
rhohair=187;             %Area density of her hair (number of hairs per
centimeters squared)

%Hypothetical length of hair she starts the process with (inches)
lhair=3*2.54;            %hair length converted from inches to microns
sk=1/10^4;               %Conversion factor to make K dimensionless
K=rhohair*Dhair*sk;      %Client's hair quality factor computed

%Acceptable range of values for (lower and upper bounds);
Kd=[9751 24381]*sk; %hair quality constant based on client parameters

tdye=[15 30];
fdye=[0.0192 0.5];  %equivalent to 1/52 and 1/4, nom.25
Vdye=[29.6 98.3] ;%*1/K;
lbraid=[15.24 60.96];  %equivalent to 2*lhair  and 8*lhair
```

```
Dbraid=[0.3175 1.5785];
fbraid=[0.0192 0.25];    %equivalent to 1/52 and 1/4
ftsp=[0.000000827 0.000011574];
Tts=[3.93 4.75];
redf=0.8;  %reduction factor of quality of temperature available on home
appliances
C2s=0.5;    %0.5; %0.9
C6s=0.5;  % 0.5;
T3=100;     %(Scaling) conversion factor

%Nonlinear Inequality Constraints (Hair Styling)
g9=x(15)*x(16)-lbraid(1)*Dbraid(2)*C2s;
g10=(x(4)/T3)*x(6)-Tts(2)*ftsp(2)*C6s;

%Combined Inequality Constraints
g=[g1;g2;g3;g4;g5;g6;g7;g8;g9;g10];

%Combined Equality Constraints
h3=x(16)+(0.2802*x(11))-1.7186;
h=[h1;h2;h3];
*******************************************************************************
%Execution of Optimization Algorithm for Hair Beauty System
%Michael Alexander, Tahira Reid, Harshit Sarin
%ME 555
%April 14, 2007

clear all

format long e;

%Setting output and large scale to off
options=optimset('Display','iter','LargeScale','off','ActiveConstrTol',1e-
10,'TolCon',1e-10);

%Matrix/vectors for linear constraints (not used)
A=[]; b=[]; Aeq=[]; beq=[];

%Lower bounds nom, 0.000000827
lb=[0.05 900 0.000000207 393 1 0.000000827 11.6871 0.33 0.1 2 0.5 15 0.09123
29.6 15.24 0.3175];

%Upper bounds
ub=[0.15 1200 0.000000413 475 3 0.000011574 116.817 4 3 10 5 30 0.5 118.30
60.96 1.5785];

%Starting Point
x0=[0.05 900 0.000000207 393 1 0.000000827 11.6871 0.33 0.1 2 0.5 15 0.09123
29.6 15.24 0.3175];

%Scaling parameters for variables (currently not used)
%D=eye(16);
%s=zeros(16,1);

%for j=1:1:16
```

```
    %D(j,j)=2/(ub(j)-lb(j));
    %s(j)=-(ub(j)+lb(j))/(ub(j)-lb(j));
%end

%y0=((D*x0')+s)'

%Optimization Algorithm
[xopt,fval,exitflag,output,lambda]=fmincon('beautyFUN',x0,A,b,Aeq,beq,lb,ub,'be
autyNONLCON',options)
```